Testing for a unit root in Lee-Carter mortality model

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Abstract. Motivated by a recent discovery that the two-step inference for Lee-Carter mortality model is inconsistent when the mortality index does not follow from a nearly integrated AR(1) process, we propose a test for unit root in a Lee-Carter model with an AR(2) process for the mortality index. Although testing for a unit root has been studied extensively in econometrics, the method and asymptotic results developed in this paper are unconventional. A simulation study is conducted to examine the finite sample behavior of the proposed test.

Key words and phrases: AR process, Lee-Carter model, unit root

1 Introduction

The increased life expectancy has posted serious challenges to insurance companies and pension funds for managing longevity risk. For hedging longevity risk and pricing annuities, mortality models and their inference play an important role. Although many types of mortality models have been proposed in the literature of actuarial science, a quite popular model is the so-called Lee-Carter model, where Lee and Carter (1992) proposed to model the logarithm of the central mortality rate by

$$\log m_{x,t} = \alpha_x + \beta_x k_t + \epsilon_{x,t},$$

where $m_{x,t}$ denotes the central mortality rate for age group $x = 1, \ldots, K$ at time period $t = 1, \ldots, T$ and $\epsilon'_{x,t}s$ are random errors. Due to identification issue, conditions $\sum_{x=1}^K \beta_x = 1$ and $\sum_{t=1}^T k_t = 0$ are imposed in finding estimators $\hat{\alpha}_x, \hat{\beta}_x, \hat{k}_t$, which are obtained by the singular value decomposition method. For effective prediction, Lee and Carter (1992) further proposed to fit an ARIMA(p, d, q) model to $\hat{k}'_t s$, and found that an ARIMA$(p, 1, q)$ model is preferred in fitting existing mortality data.

Since the seminal paper of Lee and Carter (1992), many extensions and applications have appeared in the literature with an open R package ('demography'); see Brouhns, Denuit and Vermunt (2002), Li and Lee (2005), Girosi and King (2007), Cairns et al. (2011), D'amato et al (2014), Lin et al (2014), Bisetti and Favero (2014).

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Although there are wide applications of the Lee-Carter model and its extensions in policy making, there is almost no discussion on whether the two step inference procedure could lead to a correct identification of the dynamic structure of k_t, which plays a key role in forecasting mortality rates and managing longevity risks. Recently Leng and Peng (2015) unfortunately proved that the two-step inference procedure in Lee and Carter (1992) will lead to a wrong identification of the dynamics of k_t if it is not a nearly integrated AR(1) model. Therefore this raises an interesting question on how to test unit root in the Lee-Carter mortality model, where the dynamics of mortality index follows from an AR(p) model. Although testing for a unit root has studied extensively in the literature of econometrics, methods and asymptotic results developed in this paper are quite different from exiting ones; see Section 2 for details.

We organize this paper as follows. Section 2 presents the model, method and asymptotic results for testing a unit root. A simulation study and real data analysis are given in Section 3. All proofs are put in Section 4.

2 Model, methodology and asymptotic results

Consider the following Lee-Carter model:

$$\log m_{x,t} = \alpha_x + \beta_x k_t + \epsilon_{x,t}, \quad k_t = \phi_0 + \phi_1 k_{t-1} + \phi_2 k_{t-2} + e_t \tag{1}$$

for $x = 1, \ldots, K$ and $t = 1, \ldots, T$, where $\epsilon_{x,t}'s$ and $e_t's$ are independent random errors with $\mathsf{E}\,\epsilon_{x,t} = \mathsf{E}\,e_t = 0, \mathsf{E}\,\epsilon_{x,t}^2 = \sigma_x^2, \mathsf{E}\,e_t^2 = \sigma^2$. As showed in Leng and Peng (2015), the two-step estimation procedure in Lee and Carter (1992) is consistent only when $\{k_t\}$ follows from a nearly integrated AR(1) model. Therefore an interesting question is how to test $H_0 : \phi_1 = 1 \,\&\, \phi_2 = 0$ for the above Lee-Carter model. Note that $k_t's$ are unobserved and the two-step inference procedure in Lee and Carter (1992) can not be employed due to its inconsistency and unknown asymptotic behavior even if there exists one.

Rewrite (1) as

$$\log m_{x,t} = \delta_x + \phi_1 \log m_{x,t-1} + \phi_2 \log m_{x,t-2} + u_{x,t}, \tag{2}$$

where

$$u_{x,t} = \beta_x e_t + \epsilon_{x,t} - \phi_1 \epsilon_{x,t-1} - \phi_2 \epsilon_{x,t-2}. \tag{3}$$

Put $y_{x,t}^{(i)} = \log m_{x,t-i} - T^{-1} \sum_{j=1}^{T} \log m_{x,j-i}$ and $u_{x,t}^* = u_{x,t} - \frac{1}{T} \sum_{s=1}^{T} u_{x,s}$, where $\log m_{x,t}$ is defined to be zero for $t \le 0$. Therefore (2) is equivalent to

$$y_{x,t}^{(0)} = \phi_1 y_{x,t}^{(1)} + \phi_2 y_{x,t}^{(2)} + u_{x,t}^*. \tag{4}$$

This motives us to estimate ϕ_1, ϕ_2 by minimizing the following least squares

$$\sum_{x=1}^{K}\sum_{t=1}^{T}\big(y_{x,t}^{(0)}-\phi_1 y_{x,t}^{(1)}-\phi_2 y_{x,t}^{(2)}\big)^2,$$

which leads to

$$\tilde{\phi}_1=\frac{D_{0,1}D_{2,2}-D_{0,2}D_{1,2}}{D_{1,1}D_{2,2}-D_{1,2}^2}\quad\text{and}\quad\tilde{\phi}_2=\frac{D_{1,1}D_{0,2}-D_{0,1}D_{1,2}}{D_{1,1}D_{2,2}-D_{1,2}^2},$$

with $D_{i,j}=\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(i)}y_{x,t}^{(j)}$.

Although testing for unit root has received extensive study in the literature of econometrics, the study here is quite different from existing ones. First, under $H_0:\phi_1=1\,\&\,\phi_2=0$, $\log m_{x,t-1}$ is correlated with $u_{x,t}$ and thus the above least squares estimators are biased. Second, the scores $T^{-3/2}\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(1)}(y_{x,t}^{(0)}-\phi_1 y_{x,t}^{(1)}-\phi_2 y_{x,t}^{(2)})$ and $T^{-3/2}\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(2)}(y_{x,t}^{(0)}-\phi_1 y_{x,t}^{(1)}-\phi_2 y_{x,t}^{(2)})$ have a degenerate limit under $H_0:\phi_1=1\,\&\,\phi_2=0$ and nonzero trend (i.e., $\sum_{x=1}^{K}\delta_x^2>0$) by noting that the difference of the above two converges in probability to zero. To overcome the second issue, we propose to test $H_0:\phi_1=1\,\&\,\phi_1+\phi_2=1$, i.e., to use estimators for ϕ_1 and $\phi_1+\phi_2$. For dealing with the first issue, a well-known technique developed and commonly employed in econometrics is the so-called instrumental variable method. Due to the special structure of $u_{x,t}$, finding an instrumental variable is not easy at all. Instead we propose the following bias corrected least squares estimator for ϕ_1.

Write

$$\begin{aligned}&D_{0,1}D_{2,2}-D_{0,2}D_{1,2}-(D_{1,1}D_{2,2}-D_{1,2}^2)\\=\;&D_{2,2}\big[\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(1)}(y_{x,t}^{(0)}-y_{x,t}^{(1)})\big]-D_{1,2}\big[\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(2)}(y_{x,t}^{(0)}-y_{x,t}^{(1)})\big]\\=\;&-\big[\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(1)}(y_{x,t}^{(0)}-y_{x,t}^{(1)})\big]\big[\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(2)}(y_{x,t}^{(1)}-y_{x,t}^{(2)})\big]\\&+D_{1,2}\big[\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T}(y_{x,t}^{(0)}-y_{x,t}^{(1)})(y_{x,t}^{(1)}-y_{x,t}^{(2)})\big]\\=\;&I_1+I_2,\end{aligned}$$

$$\begin{aligned}&D_{1,1}D_{2,2}-D_{1,2}^2\\=\;&(D_{1,1}-D_{1,2})(D_{2,2}-D_{1,2})+D_{1,2}[D_{1,1}-D_{1,2}-(D_{1,2}-D_{2,2})]\\=\;&-\big[\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(1)}(y_{x,t}^{(1)}-y_{x,t}^{(2)})\big]\big[\sum_{x=1}^{K}\sum_{t=1}^{T}y_{x,t}^{(2)}(y_{x,t}^{(1)}-y_{x,t}^{(2)})\big]\\&+D_{1,2}\big[\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T}(y_{x,t}^{(1)}-y_{x,t}^{(2)})^2\big]\\=\;&II_1+II_2,\end{aligned}$$

and

$$
\begin{aligned}
& D_{1,1}D_{0,2} - D_{0,1}D_{1,2} \\
= \; & D_{1,1}(D_{0,2} - D_{1,2}) - D_{1,2}(D_{0,1} - D_{1,1}) \\
= \; & \big[\sum_{x=1}^K \sum_{t=1}^T y_{x,t}^{(1)}(y_{x,t}^{(1)} - y_{x,t}^{(2)})\big]\big[\sum_{x=1}^K \sum_{t=1}^T y_{x,t}^{(2)}(y_{x,t}^{(0)} - y_{x,t}^{(1)})\big] \\
& -D_{1,2}\big[\sum_{x=1}^K \sum_{t=1}^T (y_{x,t}^{(0)} - y_{x,t}^{(1)})(y_{x,t}^{(1)} - y_{x,t}^{(2)})\big] \\
= \; & III_1 - I_2.
\end{aligned}
$$

Since $\mathsf{E}\,(u_{x,t}u_{x,t-1}) \neq 0$, I_2 and II_2 have the same order and dominate I_1, II_1, III_1 under $H_0 : \phi_1 = 1 \,\&\, \phi_2 = 0$. Therefore the inconsistency of the least squares estimators is due to term I_2, and an obvious bias corrected estimator for ϕ_1 is

$$
\tilde{\phi}_1 - \frac{D_{1,2}\big[\sum_{x=1}^K \sum_{t=1}^T (y_{x,t}^{(0)} - y_{x,t}^{(1)})(y_{x,t}^{(1)} - y_{x,t}^{(2)})\big]}{D_{1,1}D_{2,2} - D_{1,2}^2}.
$$

However, after writing the term $\sum_{x=1}^K \sum_{t=1}^T y_{x,t}^{(1)}(y_{x,t}^{(0)} - y_{x,t}^{(1)})$ in I_1 as

$$
\sum_{x=1}^K \sum_{t=1}^T u_{x,t}^* y_{x,t}^{(1)} + (\phi_1 - 1)\sum_{x=1}^K \sum_{t=1}^T y_{x,t}^{(1)}(y_{x,t}^{(1)} - y_{x,t}^{(2)}) + (\phi_1 + \phi_2 - 1)\sum_{x=1}^K \sum_{t=1}^T y_{x,t}^{(1)} y_{x,t}^{(2)},
$$

which has the same limit as $\sum_{x=1}^K \sum_{t=1}^T u_{x,t}^* y_{x,t}^{(1)}$ when $\phi_1 + \phi_2 - 1 = 0$ and $\phi_1 - 1 = o(1)$, we conclude that the above bias corrected estimator can not detect the case of $\phi_1 - 1 = o(1)$ when $\phi_2 + \phi_1 - 1 = 0$, i.e., this estimator seems over-corrected. Here we propose the following bias corrected estimator for ϕ_1:

$$
\hat{\phi}_1 = \tilde{\phi}_1 - \frac{D_{1,2}\big[\sum_{x=1}^K \sum_{t=1}^T (y_{x,t}^{(0)} - y_{x,t}^{(1)})(y_{x,t}^{(1)} - y_{x,t-1}^{(2)})\big]}{D_{1,1}D_{2,2} - D_{1,2}^2}.
$$

Note that under $H_0 : \phi_1 = 1 \,\&\, \phi_2 = 0$, $\tilde{\phi}_1 + \tilde{\phi}_2$ converges in probability to one since the I_2 term disappears. In conclusion we propose to consider the joint limit of estimators $\hat{\phi}_1 - 1$ and $\tilde{\phi}_1 + \tilde{\phi}_2 - 1$ for testing $H_0 : \phi_1 = 1 \,\&\, \phi_1 + \phi_2 = 1$, which is equivalent to $H_0 : \phi_1 = 1 \,\&\, \phi_2 = 0$. Throughout we assume that

- C) $\{\epsilon_t = (\epsilon_{1,t}, , \ldots, , \epsilon_{K,t})^\tau\}_{t=1}^T$ is a sequence of independent and identically dstributed random vectors with zero mean and covariance matrix $\Sigma^\epsilon = (\sigma_{i,j}^\epsilon)$, $\{e_t\}_{t=1}^T$ is a sequence of independent and identically distributed random variables with zero mean and finite variance σ^2, and these two sequences are independent. Here A^τ denotes the transpose of vector or matrix A. Further assume $E||\epsilon_t||^\eta + E|e_t|^\eta < \infty$ for some $\eta > 2$.

Put $u_t = (u_{1,t}, , \ldots, , u_{K,t})^\tau$ and it is easy to show that, under conditions of Theorem 1 below,

$$
\begin{aligned}
& \lim_{T\to\infty} \tfrac{1}{T}\mathsf{E}\,\big\{\sum_{t=1}^T u_{i,t} \sum_{t=1}^T u_{j,t}\big\} \\
= \; & \lim_{T\to\infty} \tfrac{1}{T}\big\{\sum_{t=1}^T \mathsf{E}\,(u_{i,t}u_{j,t}) + 2\sum_{t=1}^T \mathsf{E}\,(u_{i,t}u_{j,t-1})\big\} \\
= \; & \beta_i\beta_j\sigma^2 \\
:= \; & \sigma_{i,j}.
\end{aligned}
$$

Define $\Sigma = (\sigma_{i,j})_{1\le i,j\le K}$ and $X_T(r) = \frac{1}{\sqrt{T}}\Sigma^{-1/2}\big(\sum_{t=1}^{[Tr_1]} u_{1,t},,\ldots,,\sum_{t=1}^{[Tr_K]} u_{K,t}\big)^\tau$ for $r = (r_1,,\ldots,,r_K)^\tau$. Then, like the proofs in Phillips and Durlauf (1986), we have

$$X_T(r) \stackrel{D}{\to} W(r) = \big(W_1(r_1),\ldots,W_K(r_K)\big)^\tau, \tag{5}$$

where $\stackrel{D}{\to}$ denotes convergence in space $D([0,1]^K)$ and $W_1(r_1),,\ldots,,W_K(r_K)$ are independent Brownian motions.

Throughout define $\delta = (\delta_1,\ldots,\delta_K)^\tau$, $\log m_t = (\log m_{1,t},\ldots,\log m_{K,t})^\tau$, $u_t = (u_{1,t},,\ldots,,u_{K,t})^\tau$, $y_t^{(i)} = (y_{1,t}^{(i)},\ldots,y_{K,t}^{(i)})^\tau$, $J_d(s) = W(s) + d\int_0^s e^{d(s-t)}W(t)\,dt$ and $\tilde J_d(s) = J_d(s) - \int_0^1 J_d(t)\,dt$.

Theorem 1. *Suppose model (1) holds with condition C).*

i) *If $\delta^\tau\delta = 0$, then for $\phi_1 = 1 + d_1/\sqrt{T}\,\&\,\phi_1+\phi_2 = 1 + d_2/T$, we have*

$$\big(\sqrt{T}(\hat\phi_1 - 1), T(\tilde\phi_1+\tilde\phi_2-1)\big) \stackrel{d}{\to} (Z_1, \tilde Z_1),$$

where

$$Z_1 \sim N\left(\frac{d_1\sum_{x=1}^K \sigma^\epsilon_{x,x}}{\sum_{x=1}^K(\sigma_{x,x}+2\sigma^\epsilon_{x,x})}, \frac{\sum_{x_1=1}^K\sum_{x_2=1}^K\{(\sigma_{x_1,x_2}+2\sigma^\epsilon_{x_1,x_2})^2 + 2(\sigma^\epsilon_{x_1,x_2})^2\}}{\{\sum_{x=1}^K(\sigma_{x,x}+2\sigma^\epsilon_{x,x})\}^2}\right),$$

$$\tilde Z_1 = \frac{Z_2\sum_{x=1}^K\sigma^\epsilon_{x,x}}{Z_3\sum_{x=1}^K(\sigma_{x,x}+2\sigma^\epsilon_{x,x})} + \frac{Z_2+\sum_{x=1}^K\sigma^\epsilon_{x,x}}{Z_3} + d_2\frac{\sum_{x=1}^K(\sigma_{x,x}+3\sigma^\epsilon_{x,x})}{\sum_{x=1}^K(\sigma_{x,x}+2\sigma^\epsilon_{x,x})},$$

$$Z_2 = \frac{1}{2}tr\left(\Sigma\big(\tilde J_{d_2}(1)\tilde J_{d_2}^\tau(1) - \tilde J_{d_2}(0)\tilde J_{d_2}^\tau(0) - 2(d_2+d_1^2)\int_0^1\tilde J_{d_2}(s)\tilde J_{d_2}^\tau(s)\,ds\big) - \Sigma - 2\Sigma^\epsilon\right),$$

$Z_3 = tr\left(\Sigma\int_0^1\tilde J_{d_2}(s)\tilde J_{d_2}^\tau(s)\,ds\right)$ *and Z_1 is independent of $W(s)$, where $tr(A)$ denotes the trace of matrix A and $\stackrel{d}{\to}$ denotes the convergence in distribution.*

ii) *If $\delta^\tau\delta > 0$, then for $\phi_1 = 1 + d_1/\sqrt{T}\,\&\,\phi_1+\phi_2 = 1+d_2/T^{3/2}$, we have*

$$\big(\sqrt{T}(\hat\phi_1-1), T^{3/2}(\tilde\phi_1+\tilde\phi_2-1)\big) \stackrel{d}{\to} (Z_1, \tilde Z_1^*),$$

where

$$\tilde Z_1^* = \frac{12\sum_{x=1}^K(\sigma_{x,x}+3\sigma^\epsilon_{x,x})}{\delta^\tau\delta\sum_{x=1}^K(\sigma_{x,x}+2\sigma^\epsilon_{x,x})}Z_4 + d_2\frac{\sum_{x=1}^K(\sigma_{x,x}+3\sigma^\epsilon_{x,x})}{\sum_{x=1}^K(\sigma_{x,x}+2\sigma^\epsilon_{x,x})}$$

and

$$Z_4 = tr\left(\Sigma^{1/2}\left(\frac{W(1)}{2} - \int_0^1 W(s)\,ds\right)\delta^\tau\right) \sim N\big(0, \frac{1}{12}tr(\Sigma\delta\delta^\tau)\big).$$

For mortality rates, one usually has $\delta^\tau\delta > 0$ in practice. Hence Theorem 1ii) motivates the following test statistic for testing $H_0: \phi_1 = 1\,\&\,\phi_2 = 0$.

Define

$$\hat\delta_x = \frac{1}{T}\log m_{x,T}, \quad \hat\delta = (\hat\delta_1,\ldots,\hat\delta_K)^\tau, \quad \hat u_t^* = (\hat u_{1,t}^*,\ldots,\hat u_{K,t}^*)^\tau,$$

$$\hat{u}^*_{x,t} = y^{(0)}_{x,t} - \hat{\phi}_1(y^{(1)}_{x,t} - y^{(2)}_{x,t}) - (\tilde{\phi}_1 + \tilde{\phi}_2)y^{(2)}_{x,t},$$

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}\hat{u}^*_t\hat{u}^{*\tau}_t + \frac{2}{T}\sum_{t=1}^{T}\hat{u}^*_t\hat{u}^{*\tau}_{t-1}, \quad \hat{\Delta}_1 = tr(\hat{\Sigma}), \quad \hat{\Sigma}^{\epsilon} = -\frac{1}{T}\sum_{t=1}^{T}\hat{u}^*_t\hat{u}^{*\tau}_{t-1},$$

$$\hat{\Delta}_2 = tr(\hat{\Sigma}^{\epsilon}) \quad \text{and} \quad \hat{\Delta}_3 = sum\big((\hat{\Sigma} + 2\hat{\Sigma}^{\epsilon})^2 + 2(\hat{\Sigma}^{\epsilon})^2\big),$$

where $sum(A)$ means summation of all elements in the matrix A. Then our test statistic is defined as

$$Z = \frac{T(\hat{\phi}_1 - 1)^2(\hat{\Delta}_1 + 2\hat{\Delta}_2)^2}{\hat{\Delta}_3} + \frac{T^3(\tilde{\phi}_1 + \tilde{\phi}_2 - 1)^2(\hat{\delta}^{\tau}\hat{\delta})^2(\hat{\Delta}_1 + 2\hat{\Delta}_2)^2}{12tr(\hat{\Sigma}\hat{\delta}\hat{\delta}^{\tau})(\hat{\Delta}_1 + 3\hat{\Delta}_2)^2}. \tag{6}$$

It immediately follows from Theorem 1ii) that Z has a chi-squared limiting distribution with two degrees of freedom under $H_0 : \phi_1 = 1 \,\&\, \phi_2 = 0$ as $T \to \infty$ when all $\delta_x \neq 0$. Hence we reject H_0 at level α if $Z > \chi^2_{1-\alpha,2}$, where $\chi^2_{1-\alpha,2}$ denotes the $(1-\alpha)$-th quantile of a chi-squared distribution with two degrees of freedom.

3 Data analysis and simulation study

First we apply the proposed test in Section 2 for testing $H_0 : \phi_1 = 1 \,\&\, \phi_2 = 0$ in model (1) to the US mortality rate data from year 1933 to year 2010 with two different age groups by assuming $\delta^T\delta > 0$. Group I contains $K = 24$ age groups, which are $0, 1{-}4, 5{-}9, 10{-}14, \ldots, 105{-}109, 110+$, and Group II represents $K = 12$ age groups, which are $10-14, 15-19, \ldots, 65-69$. This data set is available in the Human Mortality Database, http://www.mortality.org. Table 1 reports $\hat{\phi}_1$, $\tilde{\phi}_1 + \tilde{\phi}_2$, $\hat{\delta}^{\tau}\hat{\delta}$, the test statistic Z given in (6) and its P-value, which rejects the null hypothesis, i.e., applying the two-step inference in Lee and Carter (1992) to the US mortality rates is problematic. We also plot $\hat{\delta}$, $diag(\hat{\Sigma})$ and $diag(\hat{\Sigma}^{\epsilon})$ in Figure 1, which are employed to set up the following simulation study.

Next we examine the finite sample performance of the proposed test by generating observations from (2) and (3) with $\phi_1 = 1 - d/\sqrt{T}, \phi_1 + \phi_2 = 1 - d/T^{3/2}$, $\delta = \hat{\delta}$, $(\beta_1, \ldots, , \beta_K)^{\tau}e_t \sim (\sqrt{\hat{\sigma}_{1,1}}, \ldots, \sqrt{\hat{\sigma}_{K,K}})^{\tau}N(0, 1)$ and $\epsilon_t \sim N\big(0, |diag(\hat{\Sigma}^{\epsilon})|\big)$, where $\hat{\delta}$, $\hat{\Sigma} = (\hat{\sigma}_{i,j})$ and $\hat{\Sigma}^{\epsilon}$ are estimators obtained from the above US mortality rates. The empirical size $(d = 0)$ and empirical power $(d = 2, 4, 6, 8)$ of the proposed test Z are computed based on 10,000 replications. Table 2 shows that the size becomes more accurate when T is larger, and is a bit larger than the nominal level for smaller T. Tables 3 and 4 show that the proposed test has nontrivial powers.

4 Proofs

Lemma 1. *Suppose conditions of Theorem 1 hold.*

i) *If* $\delta^\tau\delta = 0$*, then for* $\phi_1 = 1 + d_1/\sqrt{T}$ & $\phi_1 + \phi_2 = 1 + d_2/T$ *and any fixed* i*, we have*

- ia) $T^{-1/2} y_{[Ts]}^{(i)} \xrightarrow{D} \Sigma^{1/2} \tilde{J}_{d_2}(s)$;
- ib) $T^{-2} \sum_{t=1}^T y_t^{(i)} y_t^{(i)\tau} \xrightarrow{d} \Sigma^{1/2} \int_0^1 \tilde{J}_{d_2}(s) \tilde{J}_{d_2}^\tau(s)\, ds \Sigma^{1/2}$;
- ic) $T^{-5/2} \sum_{t=1}^T t y_t^{(i)} \xrightarrow{d} \Sigma^{1/2} \int_0^1 s \tilde{J}_{d_2}(s)\, ds$;
- id)

$$\begin{aligned} & T^{-1} \sum_{t=1}^T (y_t^{(1)} u_t^{*\tau} + u_t^* y_t^{(1)\tau}) \\ \xrightarrow{d} \; & \Sigma^{1/2} \left(\tilde{J}_{d_2}(1) \tilde{J}_{d_2}^\tau(1) - \tilde{J}_{d_2}(0) \tilde{J}_{d_2}^\tau(0) - 2(d_2 + d_1^2) \int_0^1 \tilde{J}_{d_2}(s) \tilde{J}_{d_2}^\tau(s)\, ds \right) \Sigma^{1/2} - \Sigma - 2\Sigma^\epsilon. \end{aligned}$$

ii) *If* $\delta^\tau\delta > 0$*, then for* $\phi_1 = 1 + d_1/\sqrt{T}$ & $\phi_1 + \phi_2 = 1 + d_2/T^{3/2}$ *and any fixed* i*, we have*

- iia) $T^{-3} \sum_{t=1}^T y_t^{(i)} y_t^{(i)\tau} \xrightarrow{p} \frac{1}{12} \delta\delta^\tau$;
- iib) $T^{-3} \sum_{t=1}^T t y_t^{(i)} \xrightarrow{p} \frac{1}{12} \delta$;
- iic) $T^{-3/2} \sum_{t=1}^T u_t^* y_t^{(i)\tau} \xrightarrow{d} \Sigma^{1/2} \left(\frac{W(1)}{2} - \int_0^1 W(s)\, ds \right) \delta^\tau$,

where $\xrightarrow{p}$ *denotes convergence in probability.*

Proof. Write

$$\begin{pmatrix} \log m_{x,t} \\ \log m_{x,t-1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \log m_{x,t-1} \\ \log m_{x,t-2} \end{pmatrix} + \begin{pmatrix} \delta_x + u_{x,t} \\ 0 \end{pmatrix},$$

and it follows from iterations that

$$\begin{pmatrix} \log m_{x,t} \\ \log m_{x,t-1} \end{pmatrix} = \sum_{i=1}^t \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}^{t-i} \begin{pmatrix} \delta_x + u_{x,i} \\ 0 \end{pmatrix}.$$

Using the arguments in Kölbl (2006), we have

$$\log m_{x,t} = \sum_{i=1}^{t-1} \frac{\lambda_1^{i+1} - \lambda_2^{i+1}}{\lambda_1 - \lambda_2} (\delta_x + u_{x,t-i}), \tag{7}$$

where

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad \text{and} \quad \lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}.$$

First we consider the case of $\delta^\tau\delta = 0$. In this case,

$$\lambda_1 = 1 + \frac{d_2}{T} + o(T^{-1}) \quad \text{and} \quad \lambda_2 = \frac{d_1}{\sqrt{T}} + o(T^{-1/2}). \tag{8}$$

It follows from (7), (8), (5) and the same arguments in Phillips (1987) that

$$\frac{\Sigma^{-1/2}}{\sqrt{T}} \log m_{[Ts]} \xrightarrow{D} \int_0^s e^{d_2(s-t)}\, dW(t) = J_{d_2}(s),$$

i.e.,

$$\frac{\Sigma^{-1/2}}{\sqrt{T}} y^{(0)}_{[Ts]} \xrightarrow{D} J_{d_2}(s) - \int_0^1 J_{d_2}(t)\,dt = \tilde{J}_{d_2}(s). \tag{9}$$

Hence ia)–ic) follows from (9) easily. For proving id), it follows from (4) that

$$y^{(0)}_t = \phi_1 y^{(1)}_t + \phi_2 y^{(2)}_t + u^*_t. \tag{10}$$

So we have

$$\begin{aligned}
& \tfrac{1}{T}\textstyle\sum_{t=1}^T y_t y_t^\tau \\
= \; & \tfrac{1}{T}\textstyle\sum_{t=1}^T (\phi_1 y^{(1)}_t + \phi_2 y^{(2)}_t + u^*_t)(\phi_1 y^{(1)}_t + \phi_2 y^{(2)}_t + u^*_t)^\tau \\
= \; & \tfrac{1}{T}\textstyle\sum_{t=1}^T \phi_1^2 y^{(1)}_t y^{(1)\tau}_t + \tfrac{1}{T}\sum_{t=1}^T \phi_2^2 y^{(2)}_t y^{(2)\tau}_t + \tfrac{1}{T}\sum_{t=1}^T u^*_t u^{*\tau}_t \\
& + \tfrac{1}{T}\textstyle\sum_{t=1}^T \phi_1\phi_2 (y^{(1)}_t y^{(2)\tau}_t + y^{(2)}_t y^{(1)\tau}_t) + \tfrac{1}{T}\sum_{t=1}^T \phi_1 (y^{(1)}_t u^{*\tau}_t + u^*_t y^{(1)\tau}_t) \\
& + \tfrac{1}{T}\textstyle\sum_{t=1}^T \phi_2 (y^{(2)}_t u^{*\tau}_t + u^*_t y^{(2)\tau}_t) \\
= \; & \tfrac{1}{T}\textstyle\sum_{t=1}^T (1 + \frac{2d_1}{\sqrt{T}} + \frac{d_1^2}{T}) y^{(1)}_t y^{(1)\tau}_t + \tfrac{1}{T}\sum_{t=1}^T \frac{d_1^2}{T} y^{(2)}_t y^{(2)\tau}_t + \tfrac{1}{T}\sum_{t=1}^T u^*_t u^{*\tau}_t \\
& + \tfrac{1}{T}\textstyle\sum_{t=1}^T (\frac{d_2}{T} - \frac{d_1}{\sqrt{T}} - \frac{d_1^2}{T})(y^{(1)}_t y^{(2)\tau}_t + y^{(2)}_t y^{(1)\tau}_t) \\
& + \tfrac{1}{T}\textstyle\sum_{t=1}^T (y^{(1)}_t u^{*\tau}_t + u^*_t y^{(1)\tau}_t)(1 + o_p(1)) + o_p(1) \\
= \; & \tfrac{1}{T}\textstyle\sum_{t=1}^T y^{(1)}_t y^{(1)\tau}_t + \tfrac{1}{T}\sum_{t=1}^T (\frac{2d_1}{\sqrt{T}} + \frac{d_1^2}{T})(1 + \frac{d_1}{\sqrt{T}})^2 y^{(2)}_t y^{(2)\tau}_t \\
& + \tfrac{1}{T}\textstyle\sum_{t=1}^T \frac{d_1^2}{T} y^{(2)}_t y^{(2)\tau}_t + \tfrac{1}{T}\sum_{t=1}^T u^*_t u^{*\tau}_t \\
& + \tfrac{1}{T}\textstyle\sum_{t=1}^T (\frac{d_2}{T} - \frac{d_1}{\sqrt{T}} - \frac{d_1^2}{T})(1 + \frac{d_1}{\sqrt{T}})(y^{(2)}_t y^{(2)\tau}_t + y^{(2)}_t y^{(2)\tau}_t) \\
& + \tfrac{1}{T}\textstyle\sum_{t=1}^T (y^{(1)}_t u^{*\tau}_t + u^*_t y^{(1)\tau}_t)(1 + o_p(1)) + o_p(1) \\
= \; & \tfrac{1}{T}\textstyle\sum_{t=1}^T y^{(1)}_t y^{(1)\tau}_t + \tfrac{1}{T}\sum_{t=1}^T \frac{2(d_1^2 + d_2)}{T} y^{(2)}_t y^{(2)\tau}_t \\
& + \tfrac{1}{T}\textstyle\sum_{t=1}^T u^*_t u^{*\tau}_t + \tfrac{1}{T}\sum_{t=1}^T (y^{(1)}_t u^{*\tau}_t + u^*_t y^{(1)\tau}_t)(1 + o_p(1)) + o_p(1),
\end{aligned}$$

which implies id) by using ia), ib) and $T^{-1}\sum_{t=1}^T u^*_t u^{*\tau}_t \xrightarrow{p} \Sigma + 2\Sigma^\epsilon$.

Next we consider the case of $\delta^\tau\delta > 0$. In this case,

$$\lambda_1 = 1 + \frac{d_2}{T^{3/2}} + o(T^{-3/2}) \quad \text{and} \quad \lambda_2 = \frac{d_1}{\sqrt{T}} + o(T^{-1/2}). \tag{11}$$

Then it follows from (7) and (11) that

$$T^{-1} \log m_{[Ts]} \xrightarrow{p} s\delta,$$

i.e.,

$$T^{-1} y^{(0)}_{[Ts]} \xrightarrow{p} \left(s - \int_0^1 y\,dy\right)\delta = (s - 1/2)\delta. \tag{12}$$

Hence we can show iia)–iic) by using (12). □

Proof of Theorem 1. Define

$$A_{i,j} := \sum_{x=1}^{K}\sum_{t=1}^{T}(y_{x,t-i}^{(0)} - \phi_1 y_{x,t-i}^{(1)} - \phi_2 y_{x,t-i}^{(2)})(y_{x,t-j}^{(0)} - \phi_1 y_{x,t-j}^{(1)} - \phi_2 y_{x,t-j}^{(2)}) = \sum_{x=1}^{K}\sum_{t=1}^{T} u_{x,t-i}^{*} u_{x,t-j}^{*}$$

and

$$B_{i,j} := \sum_{x=1}^{K}\sum_{t=1}^{T} y_{x,t}^{(j)}(y_{x,t-i}^{(0)} - \phi_1 y_{x,t-i}^{(1)} - \phi_2 y_{x,t-i}^{(2)}) = \sum_{x=1}^{K}\sum_{t=1}^{T} u_{x,t-i}^{*} y_{x,t}^{(j)}.$$

i) It is easy to show that

$$\begin{aligned} T^{-1}A_{i,j} &= \textstyle\sum_{x=1}^{K} T^{-1}\sum_{t=1}^{T} u_{x,t-i}^{*}u_{x,t-j}^{*} \\ &\xrightarrow{p} \textstyle\sum_{x=1}^{K} \mathsf{E}\,(\beta_x e_{t-i} + \epsilon_{x,t-i} - \epsilon_{x,t-i-1})(\beta_x e_{t-j} + \epsilon_{x,t-j} - \epsilon_{x,t-j-1}) \\ &= \begin{cases} \sum_{x=1}^{K}(\sigma_{x,x} + 2\sigma_{x,x}^{\epsilon}) & \text{if } \quad i = j \\ -\sum_{x=1}^{K}\sigma_{x,x}^{\epsilon} & \text{if } \quad |i-j| = 1 \\ 0 & \text{if } \quad |i-j| > 1, \end{cases} \end{aligned} \tag{13}$$

$$\begin{aligned} & \lim\nolimits_{T\to\infty} T^{-1}\mathsf{E}\left\{\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T} u_{x,t}^{*}u_{x,t-2}^{*}\right\}^2 \\ = \; & \lim\nolimits_{T\to\infty} T^{-1}\mathsf{E}\left\{\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T} u_{x,t}u_{x,t-2}\right\}^2 \\ = \; & \lim\nolimits_{T\to\infty} \textstyle\sum_{x_1=1}^{K}\sum_{x_2=1}^{K}\frac{1}{T}\sum_{t=1}^{T}\left\{\mathsf{E}\,(u_{x_1,t}u_{x_2,t})\right\}^2 \\ & + \lim\nolimits_{T\to\infty} \textstyle\sum_{x_1=1}^{K}\sum_{x_2=1}^{K}\frac{2}{T}\sum_{t=1}^{T}\mathsf{E}\,(u_{x_1,t}u_{x_1,t-2}u_{x_2,t-1}u_{x_2,t-3}) \\ = \; & \textstyle\sum_{x_1=1}^{K}\sum_{x_2=1}^{K}\left[(\sigma_{x_1,x_2} + 2\sigma_{x_1,x_2}^{\epsilon})^2 + 2(\sigma_{x_1,x_2}^{\epsilon})^2\right] \end{aligned}$$

and

$$\frac{1}{\sqrt{T}}A_{0,2} \xrightarrow{d} N\left(0, \sum_{x_1=1}^{K}\sum_{x_2=1}^{K}\left[(\sigma_{x_1,x_2} + 2\sigma_{x_1,x_2}^{\epsilon})^2 + 2(\sigma_{x_1,x_2}^{\epsilon})^2\right]\right), \tag{14}$$

which is independent of $W(s)$. It follows from Lemma 1 i) and (13) that

$$T^{-2}D_{i,i} = tr\left(T^{-2}\sum_{t=1}^{T} y_t^{(i)}y_t^{(i)\tau}\right) \xrightarrow{d} tr\left(\Sigma\int_0^1 \tilde{J}_{d_2}(s)\tilde{J}_{d_2}^{\tau}(s)\,ds\right), \tag{15}$$

$$\begin{aligned} T^{-1}B_{i,i+1} &= tr(T^{-1}\textstyle\sum_{t=1}^{T} u_{t-i}^{*}y_t^{(i+1)\tau}) \\ &= \tfrac{1}{2}tr\big(T^{-1}\textstyle\sum_{t=1}^{T}(u_{t-i}^{*}y_t^{(i+1)\tau} + y_t^{(i+1)}u_{t-i}^{*\tau})\big) \\ &\xrightarrow{d} \tfrac{1}{2}tr\left(\Sigma\big(\tilde{J}_{d_2}(1)\tilde{J}_{d_2}^{\tau}(1) - \tilde{J}_{d_2}(0)\tilde{J}_{d_2}^{\tau}(0) - 2(d_2 + d_1^2)\textstyle\int_0^1 \tilde{J}_{d_2}(s)\tilde{J}_{d_2}^{\tau}(s)\,ds\big) - \Sigma - 2\Sigma^{\epsilon}\right) \\ &= Z_2, \end{aligned} \tag{16}$$

$$\begin{aligned} T^{-1}B_{i,i} &= T^{-1}\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T} u_{x,t-i}^{*}(y_{x,t}^{(i)} - y_{x,t}^{(i+1)}) + T^{-1}B_{i,i+1} \\ &\xrightarrow{d} \textstyle\sum_{x=1}^{K}(\sigma_{x,x} + 2\sigma_{x,x}^{\epsilon}) + Z_2 \end{aligned} \tag{17}$$

and

$$\begin{aligned} T^{-1}B_{i,i+2} &= T^{-1}\textstyle\sum_{x=1}^{K}\sum_{t=1}^{T} u_{x,t-i}^{*}y_{x,t}^{(i+1)} - T^{-1}\sum_{x=1}^{K}\sum_{t=1}^{T} u_{x,t-i}^{*}(y_{x,t}^{(i+1)} - y_{x,t}^{(i+2)}) \\ &\xrightarrow{d} \textstyle\sum_{x=1}^{K}\sigma_{xx}^{\epsilon} + Z_2. \end{aligned} \tag{18}$$

Now using (10), (15), (16), (13) and Lemma 1i), we have

$$\begin{aligned} & D_{0,1} - D_{1,1} \\ = \; & \sum_{x=1}^{K} \sum_{t=1}^{T} y_{x,t}^{(1)} (y_{x,t}^{(0)} - y_{x,t}^{(1)}) \\ = \; & \sum_{x=1}^{K} \sum_{t=1}^{T} u_{x,t}^{*} y_{x,t}^{(1)} + \sum_{x=1}^{K} \sum_{t=1}^{T} y_{x,t}^{(1)} \left[(\phi_1 - 1) y_{x,t}^{(1)} + \phi_2 y_{x,t}^{(2)} \right] \\ = \; & B_{0,1} + (\phi_1 - 1) \sum_{x=1}^{K} \sum_{t=1}^{T} y_{x,t}^{(1)} (y_{x,t}^{(1)} - y_{x,t}^{(2)}) + (\phi_1 + \phi_2 - 1) D_{1,2} \\ = \; & O_p(T). \end{aligned} \tag{19}$$

Similarly, we have

$$\begin{cases} D_{1,1} - D_{1,2} = & B_{1,1} + (\phi_1 - 1) \sum_{x=1}^{K} \sum_{t=1}^{T} y_{x,t}^{(1)} (y_{x,t}^{(2)} - y_{x,t}^{(3)}) \\ & +(\phi_1 + \phi_2 - 1) D_{1,3} + o_p(1) = O_p(T) \\ D_{1,2} - D_{2,2} = & B_{1,2} + (\phi_1 - 1) \sum_{x=1}^{K} \sum_{t=1}^{T} y_{x,t}^{(2)} (y_{x,t}^{(2)} - y_{x,t}^{(3)}) \\ & +(\phi_1 + \phi_2 - 1) D_{2,3} + o_p(1) = O_p(T) \\ D_{0,2} - D_{1,2} = & B_{0,2} + (\phi_1 - 1) \sum_{x=1}^{K} \sum_{t=1}^{T} y_{x,t}^{(2)} (y_{x,t}^{(1)} - y_{x,t}^{(2)}) \\ & +(\phi_1 + \phi_2 - 1) D_{2,2} + o_p(1) = O_p(T), \end{cases} \tag{20}$$

which imply that

$$D_{1,1} - D_{1,2} - (D_{1,2} - D_{2,2}) = B_{1,1} - B_{1,2} + o_p(T) = A_{1,1} + o_p(T).$$

Hence

$$\begin{aligned} & D_{1,1} D_{2,2} - D_{1,2}^2 \\ = \; & D_{1,2}[D_{1,1} - D_{1,2} - (D_{1,2} - D_{2,2})] - (D_{1,1} - D_{1,2})(D_{1,2} - D_{2,2}) \\ = \; & D_{1,2} A_{1,1} + o_p(T^3), \end{aligned} \tag{21}$$

$$\begin{aligned} & D_{0,1} D_{2,2} - D_{0,2} D_{1,2} - D_{12} \sum_{x=1}^{K} \sum_{t=1}^{T} (y_{x,t}^{(0)} - y_{x,t}^{(1)})(y_{x,t}^{(1)} - y_{x,t-1}^{(2)}) - (D_{1,1} D_{2,2} - D_{1,2}^2) \\ = \; & (D_{0,1} - D_{1,1})(D_{2,2} - D_{1,2}) - D_{1,2} \sum_{x=1}^{K} \sum_{t=1}^{T} (y_{x,t}^{(0)} - y_{x,t}^{(1)})(y_{x,t}^{(2)} - y_{x,t-1}^{(2)}) \\ = \; & O_p(T^2) - D_{12} \big[\sum_{x=1}^{K} \sum_{t=1}^{T} u_{x,t}^{*} (y_{x,t}^{(2)} - y_{x,t-1}^{(2)}) \\ & + \tfrac{d_1}{\sqrt{T}} \sum_{x=1}^{K} \sum_{t=1}^{T} (y_{x,t}^{(1)} - y_{x,t}^{(2)})(y_{x,t}^{(2)} - y_{x,t-1}^{(2)}) + o_p(\sqrt{T}) \big] \\ = \; & -D_{1,2}(A_{0,2} + \tfrac{d_1}{\sqrt{T}} A_{1,2}) + o_p(T^{5/2}), \end{aligned} \tag{22}$$

$$
\begin{aligned}
& D_{0,1}D_{2,2} - D_{0,2}D_{1,2} + D_{1,1}D_{0,2} - D_{0,1}D_{1,2} - (D_{1,1}D_{2,2} - D^2_{1,2}) \\
= \;& -(D_{0,1} - D_{1,1})(D_{1,2} - D_{2,2}) + (D_{1,1} - D_{1,2})(D_{0,2} - D_{1,2}) \\
= \;& -(B_{0,1} + \tfrac{d_2}{T}D_{1,2})(B_{1,2} + \tfrac{d_2}{T}D_{2,3}) + (B_{1,1} + \tfrac{d_2}{T}D_{1,3})(B_{0,2} + \tfrac{d_2}{T}D_{2,2}) + o_p(T^2) \\
= \;& -(B_{0,1} + \tfrac{d_2}{T}D_{1,2})(B_{1,2} + \tfrac{d_2}{T}D_{1,2}) + (B_{1,1} + \tfrac{d_2}{T}D_{1,2})(B_{0,2} + \tfrac{d_2}{T}D_{1,2}) + o_p(T^2) \\
= \;& -B_{0,1}B_{1,2} + B_{1,1}B_{0,2} + \tfrac{d_2}{T}(-B_{0,1} - B_{1,2} + B_{1,1} + B_{0,2})D_{1,2} + o_p(T^2) \\
= \;& -A_{0,1}B_{1,2} + A_{1,1}B_{0,2} + \tfrac{d_2}{T}D_{1,2}(A_{1,1} - A_{0,1}) + o_p(T^2).
\end{aligned} \tag{23}
$$

Hence Theorem 1i) follows from (21)–(23).

ii) It follows from Lemma 1 ii) that

$$
T^{-3}D_{i,j} \xrightarrow{p} \frac{\delta^\tau \delta}{12} \tag{24}
$$

and

$$
T^{-3/2}B_{i,j} \xrightarrow{d} tr\left(\Sigma^{1/2}\left(\frac{W(1)}{2} - \int_0^1 W(s)\,ds\right)\delta^\tau\right). \tag{25}
$$

As before, using (24) and (25) , we can show that

$$
D_{1,1}D_{2,2} - D^2_{1,2} = D_{1,2}A_{1,1} + o_p(T^4),
$$

$$
\begin{aligned}
& D_{0,1}D_{2,2} - D_{0,2}D_{1,2} - D_{1,2}\textstyle\sum_{x=1}^K \sum_{t=1}^T (y^{(0)}_{x,t} - y^{(1)}_{x,t})(y^{(1)}_{x,t} - y^{(2)}_{x,t-1}) - (D_{1,1}D_{2,2} - D^2_{1,2}) \\
= \;& (A_{0,2} + \tfrac{d_1}{\sqrt{T}}A_{1,2})D_{1,2} + o_p(T^{7/2}),
\end{aligned}
$$

$$
\begin{aligned}
& D_{0,1}D_{2,2} - D_{0,2}D_{1,2} + D_{1,1}D_{0,2} - D_{0,1}D_{1,2} - (D_{1,1}D_{2,2} - D^2_{1,2}) \\
= \;& -A_{0,1}B_{1,2} + A_{1,1}B_{0,2} + \tfrac{d_2}{T^{3/2}}(A_{1,1} - A_{0,1})D_{1,2} + o_p(T^{5/2}),
\end{aligned}
$$

which imply Theorem 1ii) by noting that

$$
\mathsf{E}\left(\frac{W_i(1)}{2} - \int_0^1 W_i(s)\,ds\right)^2 = \frac{1}{4} - \int_0^1 s\,ds + \int_0^1\int_0^1 \min(s,t)\,dsdt = \frac{1}{4} - \frac{1}{2} + \frac{1}{3} = \frac{1}{12}
$$

and

$$
\begin{aligned}
\mathsf{E}\, Z_4^2 &= \mathsf{E}\left\{tr\left(\delta^\tau\Sigma^{1/2}\left(\tfrac{W(1)}{2} - \textstyle\int_0^1 W(s)\,ds\right)\left(\tfrac{W^\tau(1)}{2} - \textstyle\int_0^1 W^\tau(s)\,ds\right)\Sigma^{1/2}\delta\right)\right\} \\
&= tr\left(\mathsf{E}\left[\left(\tfrac{W(1)}{2} - \textstyle\int_0^1 W(s)\,ds\right)\left(\tfrac{W^\tau(1)}{2} - \textstyle\int_0^1 W^\tau(s)\,ds\right)\right]\Sigma^{1/2}\delta\delta^\tau\Sigma^{1/2}\right) \\
&= \tfrac{1}{12}tr\left(I_{K\times K}\Sigma^{1/2}\delta\delta^\tau\Sigma^{1/2}\right) \\
&= \tfrac{1}{12}tr\left(\Sigma\delta\delta^\tau\right).
\end{aligned}
$$

□

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Table 1: *US Mortality Rates.* We report estimators $\hat{\phi}_1$, $\tilde{\phi}_1 + \tilde{\phi}_2$, $\hat{\delta}^\tau \hat{\delta}$ for ϕ_1, $\phi_1 + \phi_2$ and $\delta^\tau \delta$, respectively. The computed test statistic Z defined in (6) and its P-value are reported too. Group I means age groups $0, 1-4, 5-9, \cdots, 105-109, 110+$, and Group II means age groups $10-14, \cdots, 65-69$.

	Group I	Group II
$\hat{\phi}_1$	1.0301	1.0674
$\tilde{\phi}_1 + \tilde{\phi}_2$	0.9519	0.9829
$\hat{\delta}^\tau \hat{\delta}$	0.1161	0.080
Z	454	138
P-value	0	0

Table 2: *Empirical size.* We compute the empirical size of the proposed test Z at nominal level $\alpha = 0.05, 0.1$ based on 10,000 repetitions for size $T = 50, 100, 200, 500$ from (2) and (3) with $\phi_1 = 1 \,\&\, \phi_2 = 0$.

	$\alpha = 0.05$				$\alpha = 0.1$			
	T=50	T=100	T=200	T=500	T=50	T=100	T=200	T=500
Group I	0.1483	0.0951	0.0692	0.0579	0.2280	0.1583	0.1311	0.1135
Group II	0.0912	0.0625	0.0578	0.0532	0.1453	0.1155	0.1114	0.1062

Table 3: *Empirical power for Group I.* We report the empirical power of the proposed test Z at nominal level $\alpha = 0.05, 0.1$ based on 10,000 repetitions for size $T = 50, 100, 200, 500$ from (2) and (3) with $\phi_1 = 1 - d/\sqrt{T} \,\&\, \phi_2 = d/\sqrt{T} - d/T^{3/2}$.

	$\alpha = 0.05$				$\alpha = 0.1$			
d	T=50	T=100	T=200	T=500	T=50	T=100	T=200	T=500
2	0.4804	0.3385	0.2478	0.1931	0.5846	0.4472	0.3513	0.2966
4	0.7113	0.6868	0.5534	0.4740	0.7515	0.7665	0.6589	0.5908
6	0.6803	0.7074	0.8404	0.7499	0.6917	0.7202	0.8867	0.8323
8	1.000	0.5300	0.8008	0.9253	1.000	0.5442	0.8062	0.9557

Table 4: *Empirical power for Group II.* We report the empirical power of the proposed test Z at nominal level $\alpha = 0.05, 0.1$ based on $10,000$ repetitions for size $T = 50, 100, 200, 500$ from (2) and (3) with $\phi_1 = 1 - d/\sqrt{T}$ & $\phi_2 = d/\sqrt{T} - d/T^{3/2}$.

	$\alpha = 0.05$				$\alpha = 0.1$			
d	T=50	T=100	T=200	T=500	T=50	T=100	T=200	T=500
2	0.2866	0.2236	0.2089	0.2000	0.3727	0.3226	0.3092	0.2983
4	0.5242	0.6047	0.5541	0.5437	0.5802	0.6961	0.6598	0.6631
6	0.6252	0.7027	0.8760	0.8488	0.6554	0.7236	0.9220	0.9061
8	1.000	0.6081	0.8961	0.9792	1.000	0.6258	0.8993	0.9883

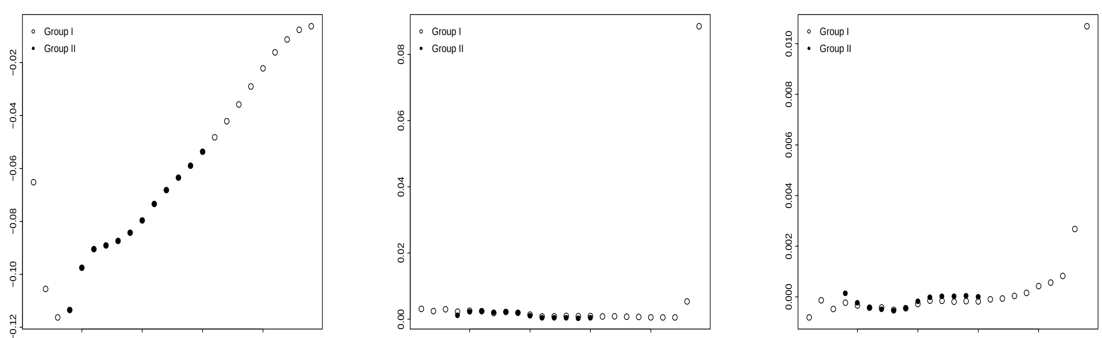

Figure 1: Estimators $\hat{\delta}$, $diag(\hat{\Sigma})$ and $diag(\hat{\Sigma}^{\epsilon})$ are plotted for the US mortality rates.