A New Approach to Identifying the Real Effects of Uncertainty Shocks

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Abstract

This paper proposes a multivariate stochastic volatility-in-vector autoregression model called the conditional autoregressive inverse Wishart-in-VAR (CAIW-in-VAR) model as a framework for studying the real effects of uncertainty shocks. We make three contributions to the literature. First, the uncertainty shocks we analyze are estimated directly from macroeconomic data so they are associated with changes in the volatility of the shocks hitting the macroeconomy. Second, we advance a new approach to identify uncertainty shocks by placing limited economic restrictions on the first and second moment responses to these shocks. Third, we consider an extension of the sign restrictions methodology of Uhlig (2005) to uncertainty shocks. To illustrate our methods, we ask what is the role of financial markets in transmitting uncertainty shocks to the real economy? We find evidence that an increase in uncertainty leads to a decline in industrial production only if associated with a deterioration in financial conditions.

Key words: Uncertainty, vector autoregression, volatility-in-mean, Wishart process, multivariate stochastic volatility.

JEL codes: C11, C32, E32

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[1]New version of the draft will be available at: sites.google.com/site/molinzhong or minchulshin.com.

1 Introduction

What are the real effects of uncertainty on the macroeconomy? This question has been challenging to analyze empirically because uncertainty is not observed. To tackle this empirical problem, our paper takes the following approach. First, we propose a flexible econometric model with stochastic volatility that allows for the volatility-in-mean effect. We then show how to impose restrictions on the first and second moment responses to an uncertainty shock of interest. These response restrictions could come from a priori theorizing and shape the class of uncertainty shocks the researcher is investigating. Finally, we discuss how to estimate and analyze the model.

We propose a multivariate stochastic volatility-in-vector autoregression model called the conditional autoregressive inverse Wishart-in-vector autoregression (CAIW-in-VAR) model. The model allows for a first-order effect of stochastic volatility, which gives a framework to estimate and evaluate the real effects of uncertainty shocks. Our framework is especially appropriate for a situation in which the researcher would like to impose economic restrictions on the responses to the uncertainty shocks of interest. These can be in terms of the expected responses of the observed economic variables to the uncertainty shocks, which we call first moment restrictions, or the expected stochastic covariance responses, which we call second moment restrictions. Our methodology can handle sign restrictions on either the first or second moment responses, or both[2].

Our new strategy for identifying the real effects of uncertainty shocks closely connects with our conditional autoregressive inverse Wishart volatility process. This volatility process, introduced into the financial econometrics literature by Golosnoy et al. (2012) and in macroeconomics by Karapanagiotidis (2012), models time-varying volatility with the Wishart family of distributions (See Philipov and Glickman, 2006; Gourieroux et al., 2009; Fox and West, 2013, for alternative autoregressive Wishart models.). The structure of the volatility dynamics makes it such that the process can be written in a linear vector autoregressive form with innovations that are martingale difference sequences. Upon writing the volatility process in vector autoregressive form, it is possible to impose restrictions directly on the responses to the volatility shocks without having to consider any explicit restrictions on the first moment shocks. We emphasize, however, that our identification methodology does not require any specific volatility process and is in fact amenable to many different reduced-form processes. We use the conditional autoregressive inverse Wishart volatility process for its

[2]Sign restrictions impose conditions on the signs of the responses to an uncertainty shock at various horizons.

flexibility and attractive property that its estimation results do not depend on the ordering of the observable variables in the system.

We develop a Markov chain Monte Carlo algorithm relying on a Metropolis-within-Gibbs sampler to simulate from the posterior density. The algorithm builds off of the one developed in Philipov and Glickman (2006) and Rinnergschwentner et al. (2011). It is complicated by the fact that the stochastic volatility process also enters into the conditional mean equation, which the previous papers do not consider. In analyzing the estimated model, we also show how to construct impulse response functions to the uncertainty shocks of interest.

Our approach has three main novelties relative to the current literature. First, we have an econometric model allowing for time-varying second moments, so we estimate uncertainty directly from the macroeconomic data. This is unlike the literature that uses proxies such as the VIX or Economic Policy Uncertainty Index to identify uncertainty shocks (Bloom, 2009; Baker et al., 2013; Scotti, 2013) and is closer to the approach taken in Jurado et al. (2015). One criticism of the volatility proxy approach is that it is unclear what the uncertainty proxies truly capture[3]. With our approach, we do not run into this interpretation issue. Relative to Jurado et al. (2015), who first extract common fluctuations in stochastic volatility from a large panel of macroeconomic and financial variables and then run a VAR on macroeconomic variables, we have a one-step estimation procedure and identify different sources of uncertainty shocks through economic restrictions.

Second, the form of our volatility process allows us to put economic restrictions directly on the responses to the uncertainty shocks of interest. Specifically, we can place restrictions on the expected responses of the first moments and second moments to uncertainty shocks. Previous approaches in the literature to investigate the real effects of uncertainty shocks with a stochastic volatility-in-vector autoregression model, such as Mumtaz and Zanetti (2013), Creal and Wu (2014), Jo (2014), and Montes-Galdon (2015), initially identify first moment structural shocks and then put stochastic volatility on those shocks. Our complementary approach is arguably more appropriate when the researcher would like to restrict the uncertainty shock under consideration through its first and second moment responses and does not want to risk further misspecification. Importantly, the researcher can be explicit about the conditions imposed without fully identifying the entire set of first and second moment shocks.

[3]For instance, it is unclear whether VIX measures macroeconomic uncertainty or just uncertainty in the financial sector. Moreover, these approaches do not allow the volatility proxies to change the forecast error variance of economic variables in the VAR system, which might not be consistent with the role of uncertainty in the economy.

Third, we consider an extension of the sign restrictions approach of Uhlig (2005) and Arias et al. (2014) to uncertainty shocks. Sign restrictions allow the researcher to impose directional responses on a set of first and second moments to uncertainty shocks at various horizons while remaining agnostic about the responses outside of the set. The previous literature on the real effects of uncertainty shocks has not considered this identification strategy.[4] An example of a first moment restriction is that an uncertainty shock transmitted through the financial sector worsens a financial conditions indicator in expectation for a certain horizon. An example of a second moment restriction is that an uncertainty shock increases the conditional variance of the innovations to all variables in the economy in expectation for a certain horizon.

As an illustration of our empirical framework, we investigate the relationship between the financial market and uncertainty shocks. We estimate a $4-$variable version of our model on monthly data with industrial production, the consumer price index, the federal funds rate, and the excess bond premium (EBP) of Gilchrist and Zakrajsek (2012). We use two different strategies to identify the real effects of an uncertainty shock. In the first case, we impose only that uncertainty increases in the economy, which is a contemporaneous second moment sign restriction that the conditional variances of all shocks hitting the economy increase. There is some evidence of a decline in industrial production and the price level following an uncertainty shock, but the responses are not significant. Crucially, there does not seem to be any significant change in financial conditions following the uncertainty shock. In our second exercise, we further impose a multi-step first moment sign restriction that financial conditions worsen in expectation for 3 months after the uncertainty shock. Upon doing so, we find that the uncertainty shock leads to a significant decline in industrial production for 15 months. The response is hump-shaped with a maximal posterior median decline of around -0.2% following a 1 standard deviation uncertainty shock. These results complement the findings of Caldara et al. (2013) and Ferreira (2014), who find that the financial channel is important in transmitting uncertainty shocks.

Our work speaks to three main literatures. First, we contribute to the empirical literature investigating the real effects of uncertainty shocks. One strand of the literature, as mentioned already, uses identified innovations to volatility proxies in vector autoregressions to identify macroeconomic movements from uncertainty shocks. The second strand of literature, also previously addressed, uses the stochastic volatility-in-vector autoregression model to analyze the real effects of uncertainty shocks. The third strand of literature, exemplified by Fernandez-Villaverde et al. (2011) and Fernandez-Villaverde et al. (2013), first estimates

[4]There are papers that utilize the sign restriction identification approach to study the real effect of uncertainty shocks in conjunction with volatility proxies (Ferreira, 2014). However, the sign restriction approach is not studied in the context of a VAR model with time-varying volatility.

the stochastic volatility from a vector autoregression and then feeds the extracted process into a dynamic equilibrium model. Relative to this work, we propose a new model that can investigate the real effects of uncertainty shocks while potentially limiting misspecification concerns via imposing limited economic restrictions. We also consider sign restriction identification of uncertainty shocks. We hope this model will become a useful tool to guide researchers in determining the important sources of the real effects of uncertainty in the economy.

The second literature with which we connect is the work on conditional heteroskedasticity-in-mean. French et al. (1987) first propose the GARCH-in-mean model in the financial economics context. Elder (2004), for inflation uncertainty, and Elder and Serletis (2010), for oil price uncertainty, use the model to investigate the volatility-in-mean effect in macroeconomic applications. Working with a stochastic volatility-in-mean model allows us to consider impulse responses to volatility shocks with no movements in the level innovations, which corresponds more closely to the notion of an uncertainty shock. Because GARCH-in-mean models do not have an independent source of variation driving volatility, they have a more cumbersome time producing these impulse responses. Koopman and Uspensky (2002) propose the univariate stochastic volatility-in-mean model. To answer many of our questions, a multivariate extension to the model is required, which we provide. Indeed, there are papers that develop and study the stochastic volatility-in-mean effect in a multivariate framework (Mumtaz and Zanetti, 2013; Creal and Wu, 2014; Jo, 2014; Montes-Galdon, 2015; Carriero et al., 2016). These works impose restrictions on level impulse responses. In this paper, we propose a novel identification strategy for the real effects of uncertainty shocks by imposing economic restrictions directly on the uncertainty shock impulse response functions.

Finally, our modeling framework builds off of a line of research modeling vector autoregressions with time-varying volatility, beginning with Uhlig (1997). Other important contributions in this field include Cogley and Sargent (2005), Primiceri (2005), and Sims and Zha (2006). These papers are concerned with whether stochastic volatility, representing changes in the nature of shocks hitting the economy, or coefficient changes, representing shifts in the underlying relationships in the economy, are more responsible for the evolving nature of macroeconomic movements in the U.S. We also consider a vector autoregression with stochastic volatility, but we allow the stochastic volatility to have a conditional mean effect as well. Therefore, while our model builds off of this literature, the questions we aim to answer are quite different. In principle, we can allow for coefficient drift as well, but for clarity of presentation, we shut off that channel.

The plan of the paper is as follows. In section 2, we present the model and discuss our framework for analyzing the real effects of uncertainty shocks. In section 3, we introduce our Markov chain Monte Carlo sampler and our algorithms to compute impulse responses. Section 4 contains our empirical application on the financial sector and uncertainty shocks and section 5 concludes.

2 Model

In this section, we lay out our proposed conditional autoregressive inverse Wishart-in-vector autoregression (CAIW-in-VAR) model. We begin by discussing our model specification. We then provide details on our volatility process and discuss the advantages of our modeling strategy. Finally, we introduce our novel identification strategy for the real effects of uncertainty shocks. We note that our framework is especially amenable to the circumstances where the researcher would like to impose only limited economic restrictions.

2.1 Model specification

We consider the following vector autoregression with multivariate stochastic volatility,

$$Y_t = \mu + \Phi Y_{t-1} + Bf(\Sigma_t) + \epsilon_t, \quad \epsilon_t|\Sigma_t \sim N(0,\ \Sigma_t) \tag{1}$$

where Y_t and μ are $k \times 1$ vectors, Φ is a $k \times k$ matrix, B is a $k \times l$ matrix (or vector), and $f(\cdot)$ is a known function that maps a $k \times k$ matrix into an $l \times 1$ vector. The forecast error ϵ_t is conditionally multivariate normal with a $k \times k$ time-varying covariance matrix Σ_t.

The term $Bf(\Sigma_t)$ captures the phenomenon called the "real effect of uncertainty shock" in macroeconomics, which allows fluctuations in the volatility of the shocks to change the conditional mean of the process. In a structural model with optimizing agents, this effect would come from agents' optimal responses to changes in risk in the economy. In our paper, we present an econometric model that can allow for these effects.

The specification of the function $f(\Sigma_t)$ is left up to the researcher as long as $f(\Sigma_t)$ enters in the conditional mean equation linearly through B. We list a few specifications for $f(\Sigma_t)$: $\log(diag(\Sigma_t)), diag(chol(\Sigma_t)), diag(\Sigma_t)$, and $vech(\Sigma_t)$. In our empirical application, we present results for $f(\Sigma_t) = \log(diag(\Sigma_t))$. We prefer this specification for the $f(\Sigma_t)$ function because it mimics the way fluctuations in uncertainty impact the first moments in a third-order perturbed dynamic equilibrium model.

Volatility process. We model the multivariate stochastic volatility with Wishart processes as in Philipov and Glickman (2006), Golosnoy et al. (2012), and Karapanagiotidis (2012),

$$\Sigma_t|\nu, S_{t-1} \sim IW(\nu, S_{t-1}^{-1}), \tag{2}$$

where $\nu > k+1$ is a scalar. The dynamics of the multivariate stochastic volatility are modeled by a $k \times k$ matrix S_t, which is defined with two additional parameter matrices C and A.

$$S_t = \frac{1}{(\nu - k - 1)}(C + A\Sigma_t A')^{-1} \tag{3}$$

C is a $k \times k$ positive definite matrix that governs the long-run mean of the multivariate volatility process. A is a $k \times k$ matrix that governs the dynamic properties of the volatility matrix process. ν is a degrees of freedom parameter that governs the conditional variance of the Σ_t random variable. This formulation ensures that the resulting scale matrix S_t is symmetric and positive definite.

Note that the process is formulated in a way that the conditional mean of the volatility matrix has the following simple form

$$E[\Sigma_t|\mathcal{F}_{t-1}] = C + A\Sigma_{t-1}A' \tag{4}$$

and

$$Cov(\Sigma_{ij,t},\ \Sigma_{lm,t}|\mathcal{F}_{t-1}) = \frac{2\Psi_{ij,t}\Psi_{lm,t} + (\nu - k + 1)(\Psi_{il,t}\Psi_{jm,t} + \Psi_{im,t}\Psi_{lj,t})}{(\nu - k)(\nu - k - 3)} \tag{5}$$

where $\mathcal{F}_{t-1} = \{\Sigma_{t-1}, \Sigma_{t-2}, ...\}$ and $\Psi_t = C + A\Sigma_{t-1}A'$. This delivers a convenient linear representation for the multivariate volatility process with innovations that are martingale difference sequences,

$$\sigma_t = \overline{C} + \overline{A}\sigma_{t-1} + \overline{v}_t, \quad E[\overline{v}_t|\mathcal{F}_{t-1}] = 0 \quad \text{and} \quad E[\overline{v}_t\overline{v}_s'|\mathcal{F}_{t-1}] = 0, \quad \forall s \neq t \tag{6}$$

where $\sigma_t = vech(\Sigma_t)$, $\overline{C} = vech(C)$ and

$$\overline{A} = L_n(A \otimes A)D_n$$

where $vec(x) = D_n vech(x)$ and $vech(x) = L_n vec(x)$.

This formulation of the volatility model allows us to clearly identify the time t shock to volatility, $\overline{v}_t$, which we call the shock to uncertainty. We focus our identifying restrictions on this shock.

In addition, the VAR form of the volatility process proves key to deriving unconditional moments and giving stationarity conditions, as discussed in Golosnoy et al. (2012).

2.2 Identification of uncertainty shocks

Our primary goal in this paper is to identify the effects of uncertainty shocks. Identifying uncertainty shocks brings along the usual challenges of identification present in the structural VAR literature. That is, an increase in one of the elements in $\overline{v}_t$ from Equation 6 (e.g., the volatility of the innovation to real activity) is potentially due to different sources (e.g., either uncertainty originating in the real economy or the financial markets). In this section, we discuss how we can distinguish various sources of fluctuations in the economic variables' forecast error variances and covariances. We call these fluctuations changes in uncertainty. Given the linear representation of the volatility process, we can focus our attention on equation 6.

We can write the time t uncertainty shock ($\overline{v}_t$) as a linear function of uncorrelated unit-variance shocks ($\overline{v}_t^*$), so we have

$$\overline{v}_t = R_t \overline{v}_t^* \quad \text{or} \quad R_t^{-1} \overline{v}_t = \overline{v}_t^*, \tag{7}$$

where R_t is a $k \times k$ invertible matrix. The conditional variance-covariance matrix of $\overline{v}_t^*$ is an identity matrix, $Var(\overline{v}_t^* | \mathcal{F}_{t-1}) = I_k$. We impose restrictions on the impulse response functions to the uncertainty shocks $\overline{v}_t^*$. Note that there is an important difference between our uncertainty shocks and structural first moment shocks (ϵ_t^*) identified from the reduced-form level shocks (ϵ_t), which have the following relationship

$$\epsilon_t = H_t \epsilon_t^* \quad \text{or} \quad H_t^{-1} \epsilon_t = \epsilon_t^*,$$

where H_t is a $k \times k$ invertible matrix. Unlike structural level shocks (ϵ_t^*), our uncertainty shocks ($\overline{v}_t^*$) have a contemporaneous impact on both the stochastic covariance matrix (Σ_t) as well as the conditional mean of the observed variables (Y_t).

It turns out that within the CAIW-in-VAR model framework, we can still utilize most of the identification methods presented in the structural VAR literature. This possibility has not been recognized so far in the literature and we view this as an advantage of our modeling framework. To see this, recall the VAR representation of the CAIW process,

$$\sigma_t = \overline{C} + \overline{A} \sigma_{t-1} + \overline{v}_t, \quad E[\overline{v}_t | \mathcal{F}_{t-1}] = 0 \quad \text{and} \quad E[\overline{v}_t \overline{v}_s' | \mathcal{F}_{t-1}] = 0, \, \forall s \neq t, \quad E[\overline{v}_t \overline{v}_t' | \mathcal{F}_{t-1}] = \Omega_t$$

where Ω_t is the conditional variance of σ_t given the information set at time $t-1$ and is a closed-form function of Σ_{t-1}. The VAR framework naturally allows us to use the identification strategies developed in the structural VAR literature to identify uncertainty shocks. It allows us to put identifying restrictions directly on the uncertainty shocks. This means that identifying the uncertainty shocks does not require explicit identification of any structural level shocks. Therefore, we can focus on identifying uncertainty shocks, which are the objects we are interested in. In this paper, we focus on the sign restrictions approach.

Another advantage of our model is that we can identify a subset of uncertainty shocks. To see this, first note that the identification of an uncertainty shock is to choose a matrix R_t that satisfies the following two conditions

$$R_t^{-1}\overline{v}_t = \overline{v}_t^* \quad \text{and} \quad R_t R_t' = \Omega_t. \tag{8}$$

where R_t is invertible for all t. Then, it is clear that to identify the ith shock in $\overline{v}_t^*$, it is only necessary to restrict elements in the ith column of R_t. This is different from previous approaches in the empirical literature investigating the real effects of uncertainty shocks where the time-varying volatilities are modeled only after the structural level shocks are fully identified.

Imposing economic restrictions on the responses to the uncertainty shocks $\overline{v}_t^*$ involves conditions on the set of impulse response functions that we consider. To fix ideas, we first define our notion of an impulse response function. We use the generalized impulse response function of Koop et al. (1996). We distinguish between first moment impulse response functions and second moment impulse response functions.

A *first moment impulse response function* gives the expected change in the conditional means of the observable variables from the jth uncertainty shock $\overline{v}_t^* = e_j$ conditional upon a previous volatility level σ_{t-1}^* (e_j is a column vector with a 1 in the jth element and zeros elsewhere). The dependence of the impulse response function on the rotation R_t is also made explicit.

$$\begin{aligned} &IRF_{t:t+S}[Y|\sigma_{t-1} = \sigma_{t-1}^*, \overline{v}_t^* = e_j; R_t] = \\ &\quad E(Y_{t:t+S}|\sigma_{t-1} = \sigma_{t-1}^*, \overline{v}_t^* = e_j; R_t) - E(Y_{t:t+S}|\sigma_{t-1} = \sigma_{t-1}^*, \overline{v}_t^* = 0; R_t) \end{aligned} \tag{9}$$

A *second moment impulse response function* gives the expected change in the variance covariance matrix of the innovations to the observable variables (ϵ_t) from the jth uncertainty shock $\overline{v}_t^* = e_j$ conditional upon a previous volatility level σ_{t-1}^*.

$$
\begin{aligned}
&IRF_{t:t+S}[\sigma|\sigma_{t-1} = \sigma_{t-1}^*, \overline{v}_t^* = e_j; R_t] = \\
&\quad E(\sigma_{t:t+S}|\sigma_{t-1} = \sigma_{t-1}^*, \overline{v}_t^* = e_j; R_t) - E(\sigma_{t:t+S}|\sigma_{t-1} = \sigma_{t-1}^*, \overline{v}_t^* = 0; R_t)
\end{aligned}
\tag{10}
$$

These impulse response functions capture the expected effect on the first and second moments of a one standard deviation movement to the uncertainty shock of interest.

In both cases, the impulse response functions are conditional upon two state variables: the time $t-1$ level of volatility σ_{t-1}^* and the uncertainty shock $\overline{v}_t^*$. In addition to being the historical level of volatility, the time $t-1$ level of volatility σ_{t-1} also impacts the variance covariance matrix Ω_t of $\overline{v}_t$. Broadly speaking, a more volatile time period will in general imply larger volatility shocks. A benchmark we use is to set $\sigma_{t-1} = E[\sigma_t]$, which is the unconditional variance. We could also investigate the impulse response functions conditional on a high volatility time period (such as the Great Recession) or a low volatility time period (such as the Great Moderation).

These facts then suggest a straightforward procedure for restricting the set of impulse response functions we consider. The economic restrictions are restrictions on the set of impulse response functions following the jth uncertainty shock. Given a value σ_{t-1}^* that the researcher fixes beforehand, these restrictions imply restrictions on the admissible set of decompositions R_t. Finding the set of R_t that satisfies the conditions completes the procedure.

We will discuss computational aspects of the impulse response functions in a later section. First, we give concrete examples of how we impose sign restrictions.

2.3 Imposing sign restrictions in a simple model

In this subsection, we illustrate our identification strategy via examples. More specifically, we consider a 2-variable CAIW-in-VAR model and illustrate two different approaches to identify the real effects of uncertainty shocks: first moment and second moment restrictions. *First moment restrictions* impose conditions on the expected responses of the observed variables (Y_t) to an uncertainty shock. They involve restrictions on the first moment impulse response functions. *Second moment restrictions* impose conditions on the expected responses of the conditional variances and covariances of the innovations to the uncertainty shock. They involve restrictions on the second moment impulse response functions. It is important to note that although our examples only consider single first and second moment restrictions, our framework can handle both types of restrictions simultaneously, or multiples of either restriction.

2-variable CAIW-in-VAR model. Let us consider a simple 2-variable example.

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \Phi \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + Bf(\Sigma_t) + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}, \quad \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} \Bigg| \underbrace{\begin{pmatrix} \Sigma_{11,t} & \Sigma_{12,t} \\ \Sigma_{12,t} & \Sigma_{22,t} \end{pmatrix}}_{\Sigma_t} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_t \right). \quad (11)$$

The volatility process is then

$$\begin{pmatrix} \Sigma_{11,t} \\ \Sigma_{12,t} \\ \Sigma_{22,t} \end{pmatrix} = \overline{C} + \overline{A} \begin{pmatrix} \Sigma_{11,t-1} \\ \Sigma_{12,t-1} \\ \Sigma_{22,t-1} \end{pmatrix} + \begin{pmatrix} \overline{v}_{11,t} \\ \overline{v}_{12,t} \\ \overline{v}_{22,t} \end{pmatrix}, \begin{pmatrix} \overline{v}_{11,t} \\ \overline{v}_{12,t} \\ \overline{v}_{22,t} \end{pmatrix} \sim \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Omega_{11,t} & \Omega_{12,t} & \Omega_{13,t} \\ \Omega_{12,t} & \Omega_{22,t} & \Omega_{23,t} \\ \Omega_{13,t} & \Omega_{23,t} & \Omega_{33,t} \end{pmatrix} \right). \quad (12)$$

Example 1: Sign restriction on the second moment. A second moment restriction puts conditions on the expected movements of the conditional variances and covariances to the uncertainty shock. For example, one can consider the following restriction

> *The conditional variance of the innovation to $y_{1,t}$ increases in response to the uncertainty shock $\overline{v}^*_{1,t}$ contemporaneously (contemporaneous second moment restriction).*

Although we present a contemporaneous condition in this example, the sign restriction can be either contemporaneous or for multiple periods. The above restriction can be written in terms of an impulse response function

$$E\left[\Sigma_{11,t+h} \Bigg| \begin{pmatrix} \Sigma_{11,t-1} \\ \Sigma_{12,t-1} \\ \Sigma_{22,t-1} \end{pmatrix} = E\left[\begin{pmatrix} \Sigma_{11,t} \\ \Sigma_{12,t} \\ \Sigma_{22,t} \end{pmatrix}\right], \begin{pmatrix} \overline{v}^*_{1,t} \\ \overline{v}^*_{2,t} \\ \overline{v}^*_{3,t} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; R_t \right] -$$
$$E\left[\Sigma_{11,t+h} \Bigg| \begin{pmatrix} \Sigma_{11,t-1} \\ \Sigma_{12,t-1} \\ \Sigma_{22,t-1} \end{pmatrix} = E\left[\begin{pmatrix} \Sigma_{11,t} \\ \Sigma_{12,t} \\ \Sigma_{22,t} \end{pmatrix}\right], \begin{pmatrix} \overline{v}^*_{1,t} \\ \overline{v}^*_{2,t} \\ \overline{v}^*_{3,t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; R_t \right] > 0, \quad \text{for } h = 0.$$

where we set the time $t-1$ value of volatility to its unconditional mean. There is no closed form for the set of admissible decompositions R_t. However, it is possible to obtain them through simulation-based methods as in Uhlig (2005) and Arias et al. (2014). Previous results in the sign restrictions literature (e.g. Uhlig, 2005) justifies us fixing $\tilde{R}_t = chol(\Omega_t)$ and checking all possible rotation matrices Q, where $QQ' = Q'Q = I_3$. This is because any decomposition R_t can be written such that

$$R_t = \tilde{R}_t Q \tag{13}$$

We keep the rotations Q that satisfy the above second moment sign restriction.

Example 2: Sign restrictions on the first moment. A first moment restriction puts conditions on the expected responses of the observable variables to an uncertainty shock. First moment restrictions could help provide a sharper identification. For example, if it is suggested from economic theory the direction of an economic variable's movement following a certain source of uncertainty shock, imposing these restrictions could shrink the candidate uncertainty shocks. In this example, we consider a multi-step first moment restriction

The expected value of $y_{1,t}$ increases in response to the uncertainty shock $\overline{v}^*_{1,t}$ for the first H periods (multi-step first moment restriction).

Again, this restriction can be written in terms of impulse response functions

$$E\left[y_{1,t+h}\middle| \begin{pmatrix}\Sigma_{11,t-1}\\ \Sigma_{12,t-1}\\ \Sigma_{22,t-1}\end{pmatrix} = E\left[\begin{pmatrix}\Sigma_{11,t}\\ \Sigma_{12,t}\\ \Sigma_{22,t}\end{pmatrix}\right], \begin{pmatrix}\overline{v}^*_{1,t}\\ \overline{v}^*_{2,t}\\ \overline{v}^*_{3,t}\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}; R_t\right] -$$
$$E\left[y_{1,t+h}\middle| \begin{pmatrix}\Sigma_{11,t-1}\\ \Sigma_{12,t-1}\\ \Sigma_{22,t-1}\end{pmatrix} = E\left[\begin{pmatrix}\Sigma_{11,t}\\ \Sigma_{12,t}\\ \Sigma_{22,t}\end{pmatrix}\right], \begin{pmatrix}\overline{v}^*_{1,t}\\ \overline{v}^*_{2,t}\\ \overline{v}^*_{3,t}\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}; R_t\right] > 0, \quad \text{for } h = 0, ..., H,$$

We can perform a simulation exercise similar to the previous example to find impulse response functions that satisfy the appropriate sign restrictions.

Sign restrictions implementation We discuss our simulation methodology to construct impulse response functions in a later section. Conditional upon being able to construct impulse response functions, implementing the sign restrictions algorithm is straightforward following the methodology of Uhlig (2005) or Arias et al. (2014). We draw the rotation matrix Q in equation 13 using a QR decomposition as in Arias et al. (2014) and keep the IRFs that satisfy the imposed sign restrictions.

2.4 Discussion on our choice of volatility process

In principle, we have multiple options for the form of our reduced-form volatility process. Modeling the reduced-form variance covariance matrix with the conditional autoregressive inverse Wishart process, however, leads to important advantages in implementing our approach. In this section, we lay out a key advantage of using our proposed volatility process in comparison to the popular model found in Primiceri (2005).

Primiceri (2005) models the volatility process Σ_t in the following fashion

$$\begin{pmatrix} \Sigma_{11,t} & \Sigma_{12,t} \\ \Sigma_{12,t} & \Sigma_{22,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha_t & 1 \end{pmatrix} \begin{pmatrix} b_{1,t} & 0 \\ 0 & b_{2,t} \end{pmatrix} \begin{pmatrix} 1 & \alpha_t \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \log b_{1,t} \\ \log b_{2,t} \\ \alpha_t \end{pmatrix} = \begin{pmatrix} \log b_{1,t-1} \\ \log b_{2,t-1} \\ \alpha_{t-1} \end{pmatrix} + \begin{pmatrix} \overline{v}^*_{\eta_{b_1},t} \\ \overline{v}^*_{\eta_{b_2},t} \\ \overline{v}^*_{\eta_\alpha,t} \end{pmatrix}, \begin{pmatrix} \overline{v}^*_{\eta_{b_1},t} \\ \overline{v}^*_{\eta_{b_2},t} \\ \overline{v}^*_{\eta_\alpha,t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \zeta^2_{\eta_{b_1}} & 0 & 0 \\ 0 & \zeta^2_{\eta_{b_2}} & 0 \\ 0 & 0 & \zeta^2_{\eta_\alpha} \end{pmatrix} \right) \tag{14}$$

The Primiceri (2005) model is attractive in some respects. For example, if one assumes a Cholesky structure of the economy, the time-varying parameters can be nicely decomposed into volatility shocks $(b_{1,t}, b_{2,t})$ and smooth changes in the structure of the economy (α_t). Under this setting, the identification of uncertainty shocks comes through the Cholesky structure and interpretation of uncertainty innovations $\overline{v}^*_{\eta_{b_1},t}$, $\overline{v}^*_{\eta_{b_2},t}$, and $\overline{v}^*_{\eta_\alpha,t}$ are readily available. If one views the volatility model as a description of volatility dynamics, however, potentially coming from a non-Cholesky model economy, this interpretation does not hold. That is, $\overline{v}^*_{\eta_{b_1},t}$, $\overline{v}^*_{\eta_{b_2},t}$, and $\overline{v}^*_{\eta_\alpha,t}$ are not structural uncertainty innovations anymore and the assumption that they are independent from each other may not be appropriate.

We can easily apply our framework to the volatility process laid out in Primiceri (2005). To study the effect of uncertainty shocks, we do not assume full identification of the VAR a priori. Instead, we use a flexible model for the forecast error variance-covariance matrix and then impose relevant restrictions directly on this volatility process to identify uncertainty shocks. One can adopt a flexible model for the volatility process in a similar way to equation

14,

$$\begin{pmatrix} \Sigma_{11,t} & \Sigma_{12,t} \\ \Sigma_{12,t} & \Sigma_{22,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha_t & 1 \end{pmatrix} \begin{pmatrix} b_{1,t} & 0 \\ 0 & b_{2,t} \end{pmatrix} \begin{pmatrix} 1 & \alpha_t \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \log b_{1,t} \\ \log b_{2,t} \\ \alpha_t \end{pmatrix} = A \begin{pmatrix} \log b_{1,t-1} \\ \log b_{2,t-1} \\ \alpha_{t-1} \end{pmatrix} + \begin{pmatrix} \overline{v}_{\eta_{b_1},t} \\ \overline{v}_{\eta_{b_2},t} \\ \overline{v}_{\eta_\alpha,t} \end{pmatrix}, \begin{pmatrix} \overline{v}_{\eta_{b_1},t} \\ \overline{v}_{\eta_{b_2},t} \\ \overline{v}_{\eta_\alpha,t} \end{pmatrix} \sim N(0, \Omega) \tag{15}$$

where A and Ω are 3×3 matrices. Matrices A and Ω reflect the fact that the Cholesky structure might not be the model that we are aiming to analyze. The Cholesky structure is simply used to model the forecast error variance-covariance matrix, Σ_t, in a reduced-form manner. Then, identification of uncertainty shocks is done by imposing restrictions on Ω in conjunction with the assumption on the relationship between $\overline{v}_t$ and $\overline{v}_t^*$, $\overline{v}_t = R\overline{v}_t^*$ as in section 2.2. Under this modeling assumption, all identification strategies that we provided in the previous section are applicable.

We prefer the CAIW model for practical reasons. As stated in Primiceri (2005), the estimated results from this decomposition are conditional upon the ordering of the variables. Therefore, in our simple example, flipping the order of $y_{1,t}$ and $y_{2,t}$ and reestimating the model will in theory lead to different results. In contrast, our conditional autoregressive inverse Wishart process does not suffer from this potential drawback.

2.5 Caveats

Even though we believe that our approach adds important tools to the macroeconomists' toolkit in analyzing the real effect of uncertainty shocks, we must also acknowledge some caveats. First, our approach does not allow level shocks to impact current uncertainty. This is an additional exclusion restriction that is implicitly assumed in our uncertainty shock identification. Within our framework, it is complicated to allow this channel because our reduced-form covariance matrices (Σ_t) need to be positive definite. In the literature, this channel can be relaxed at the cost of full identification of all structural shocks (Creal and Wu, 2014) or the use of a proxy variable[5] (Bloom, 2009). Secondly, it is hard to find a case where our model exactly maps into a fully structural model such as a dynamic equilibrium model with time-varying volatilities or a classical structural VAR. Insofar as our econometric model is flexible and the economic restrictions we impose are limited, however, we believe our

[5]Recently, Carriero et al. (2015) pointed out that proxy variables in these VARs are subject to measurement error and could potentially lead to biased parameter estimates.

toolkit can aid in the discovery of important comovements in the data that more structural models with uncertainty shocks should be able to match.

3 Bayesian Analysis of CAIW-in-VAR Models

3.1 Posterior inference

Prior specification. As we take a Bayesian perspective, the presented CAIW-in-VAR model is completed by specifying prior distributions on the unknown parameters. Parameters in the conditional mean of the model, μ, Φ and B, are assumed to follow independent multivariate normal distributions,

$$\mu \sim N\left(m_\mu, V_\mu\right), \quad vec(\Phi) \sim N\left(m_\Phi, V_\Phi\right), \quad vec(B) \sim N\left(m_B, V_B\right)$$

where $vec(\cdot)$ is the vectorize operator. The choice of this prior specification facilitates posterior computation due to its conjugacy.

There are three types of parameters in the volatility equation (A, C, and ν). The parameter A governs the dynamic properties of the volatility matrix process. Each element of A follows an independent normal distribution except the element in the far upper-left corner. The prior distribution for the $(1,1)$-th element in the A matrix is set to be a truncated normal distribution defined on the positive real line to ensure identification (see Golosnoy et al. (2012) for more details),

$$\begin{aligned} A(1,1) &\sim TN(m_{A(1,1)}, V_{A(1,1)}, 0, \infty) \\ A(i,j) &\sim N(m_{A(i,j)}, V_{A(i,j)}) \quad \forall\, (i,j) \neq (1,1). \end{aligned}$$

The parameter C determines the long-run mean of the volatility process. We set the prior for it as following an inverse Wishart distribution with scale matrix Ψ and degrees of freedom parameter df. As the Wishart-type distribution is quite a popular prior in the Bayesian literature for a variance covariance matrix, we believe it to be a natural choice for C,

$$C \sim IW\left(df, \Psi\right).$$

Finally, ν, the degrees of freedom parameter in the inverse Wishart process, follows a gamma distribution

$$\nu \sim Gamma(a_\nu, b_\nu)$$

where we truncate this distribution at $\nu > k + 3$ so that the inverse Wishart process is well defined and the variances of its elements exist.

Posterior simulator. We construct a Metropolis-within-Gibbs posterior simulator to draw from the posterior distribution of our parameters. The algorithm runs on the following cycles:

1. $p(\Sigma_t| \; others, Y)$ for $t = 1, ..., T - 1$: multivariate stochastic volatilities
2. $p(\Sigma_T| \; others, Y)$: multivariate stochastic volatility at the last period
3. $p(\mu, B, \Phi| \; others, Y)$: parameters in the conditional mean equation
4. $p(\nu| \; others, Y)$: degrees of freedom parameter in the inverse Wishart process
5. $p(C| \; others, Y)$: long-run mean parameter in the inverse Wishart process
6. $p(A| \; others, Y)$: dynamics parameter in the inverse Wishart process

where we define $p(\theta|others, Y)$ as the conditional distribution of θ given $Y_{1:T}$ and all other parameters except θ. In the appendix, we provide details of the algorithm with full conditional posterior distributions.

Our algorithm builds upon the specifications of Philipov and Glickman (2006) and Rinnergschwentner et al. (2011). Relative to their frameworks, our model has a complication in that the stochastic volatility appears in the conditional mean equation as well and therefore their posterior samplers are not directly applicable to our framework. Our algorithm adopts the single-move state simulator of Jacquier et al. (1994), which is widely used in the context of the stochastic volatility model (e.g., Cogley and Sargent, 2005; Clark, 2011, for macroeconomic applications).

3.2 Impulse response function

There are two types of impulse response functions that we consider, first moment impulse response functions and second moment impulse response functions. As we have the linear form for the volatilities in equation 6, calculating the second moment impulse response function is straightforward and no simulation methods are needed conditional upon a time $t - 1$ volatility level and time t uncertainty shock. Calculating the first moment impulse response function to an uncertainty shock is complicated by the nonlinear nature of the system. Hence, conditional upon a parameter draw, we use simulation methods to compute the impulse response function.

Continuing with our 2-variable CAIW-in-VAR example in equation 11, we present how to calculate a first moment impulse response to an uncertainty shock. An extension to a larger dimension is possible without further complication.

Algorithm 1 *(IRF of the uncertainty shock of length S in the 2-variable CAIW-in-VAR model).*

1. *Choose initial value $\tilde{\Sigma}_{-1}$ for IRF, and set $m = 1$.*
2. *Consider 1 standard deviation increase of an element in $\overline{v}_0^*$ (here we present a 1 standard deviation increase of the third element in the vector $\overline{v}_0^*$),*

$$\overline{v}_0^{*,1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \textit{versus} \quad \overline{v}_0^{*,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

3. *Form reduced-form shocks in the initial period $\overline{v}_0^1$ and $\overline{v}_0^0$. The matrix R_0 will depend on the identification scheme. As the third shock is operative, only the third column of R_0 must be identified.*

$$\overline{v}_0^1 = \begin{pmatrix} R_{13,0} \\ R_{23,0} \\ R_{33,0} \end{pmatrix} \quad \textit{versus} \quad \overline{v}_0^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

4. *Simulate two volatility paths indexed by (m) (using Equations 2 and 3) conditional on the initial shock and initial value $\tilde{\Sigma}_{-1}$,*

$$\{\Sigma_t^{(m)}(\overline{v}_0^1)\}_{t=0,\ldots,S} \quad \textit{versus} \quad \{\Sigma_t^{(m)}(\overline{v}_0^0)\}_{t=0,\ldots,S}$$

5. *For each simulated volatility path, compute $\{\overline{Y}_t^{(m)}(\overline{v}_0^{*,1})\}_{t=0,\ldots,S}$ and $\{\overline{Y}_t^{(m)}(\overline{v}_0^{*,0})\}_{t=0,\ldots,S}$ implied by the volatility paths where $\overline{Y}_t$ is a mean conditional on $\overline{Y}_{t-1}^{(m)}$ and $\Sigma_t^{(m)}$:*

$$\overline{Y}_t^{(m)} = \widehat{\mu} + \widehat{\Phi}\overline{Y}_{t-1}^{(m)} + \widehat{B} f(\Sigma_t^{(m)}).$$

   Go to step 4 with $m = m + 1$ if $m < M$; otherwise go to step 6.

6. *Form impulse response by integrating out the simulated volatility paths:*

$$IRF_{0:S}[Y_j|\tilde{\Sigma}_{-1}, \overline{v}_0^{*,1}; R_0] \approx \frac{1}{M}\sum_{m=0}^{M} \overline{Y}_{0:S}^{(m)}(\overline{v}_0^{*,1}) - \frac{1}{M}\sum_{m=0}^{M} \overline{Y}_{0:S}^{(m)}(\overline{v}_0^{*,0}).$$

Based on this algorithm, we can construct the posterior distributions for the IRFs of the uncertainty shocks. Sign-restricted IRFs require an additional simulation algorithm.

Algorithm 2 *(Posterior distribution sign-restricted IRFs).*

1. *Run the posterior sampler to obtain S draws from the posterior distribution of unknown parameters in the CAIW-in-VAR model.*
2. *For each posterior draw, θ^s, repeat the following M times,*
   (a) *Generate a candidate rotation matrix R^* by the QR decomposition.*
   (b) *Apply Algorithm 1 to obtain the impulse response function $IRF_{0:S}[Y_j|\tilde{\Sigma}_{-1}, \overline{v}_0^{*,1}; R^*]$.*[6]
3. *Among $S \times M$ IRFs, keep those that satisfy the sign restrictions.*

After running this algorithm, one obtains S^* draws of IRFs where $S^* < S \times M$. These draws form the posterior distribution of the IRFs, which can be used to construct point estimates and credible sets.

4 Empirical Application: The financial market and uncertainty shocks

Motivation. Since the Great Recession, a renewed interest has emerged on the role of the financial sector in macroeconomic fluctuations. Our empirical application investigates the relationship between the financial sector and uncertainty shocks. Heightened uncertainty could interfere with banks' willingness to lend, thereby disrupting efficient capital flows. This could be a potential channel through which the financial sector would transmit uncertainty to the real economy.

We view our empirical application as contributing to a fast growing literature on the relationship between the financial markets and the macroeconomy. Jermann and Quadrini (2012) and Christiano et al. (2014) find that shocks originating in the financial sector have significant macroeconomic effects. Fernandez-Villaverde and Rubio-Ramirez (2007) and Justiniano and Primiceri (2008) observe that volatility from shocks to the capital accumulation equation, which may be interpreted as financial sector shocks (e.g., Justiniano et al., 2011), are the main drivers of the time-varying volatility of macroeconomic fluctuations on U.S. data. Using proxies for uncertainty and financial conditions, Caldara et al. (2013) find that uncertainty shocks can lead to a significant recession only if they are transmitted through

[6] Not all R guarantee a positive definite volatility processes. We discard IRF draws that are based on non-positive definite volatility processes.

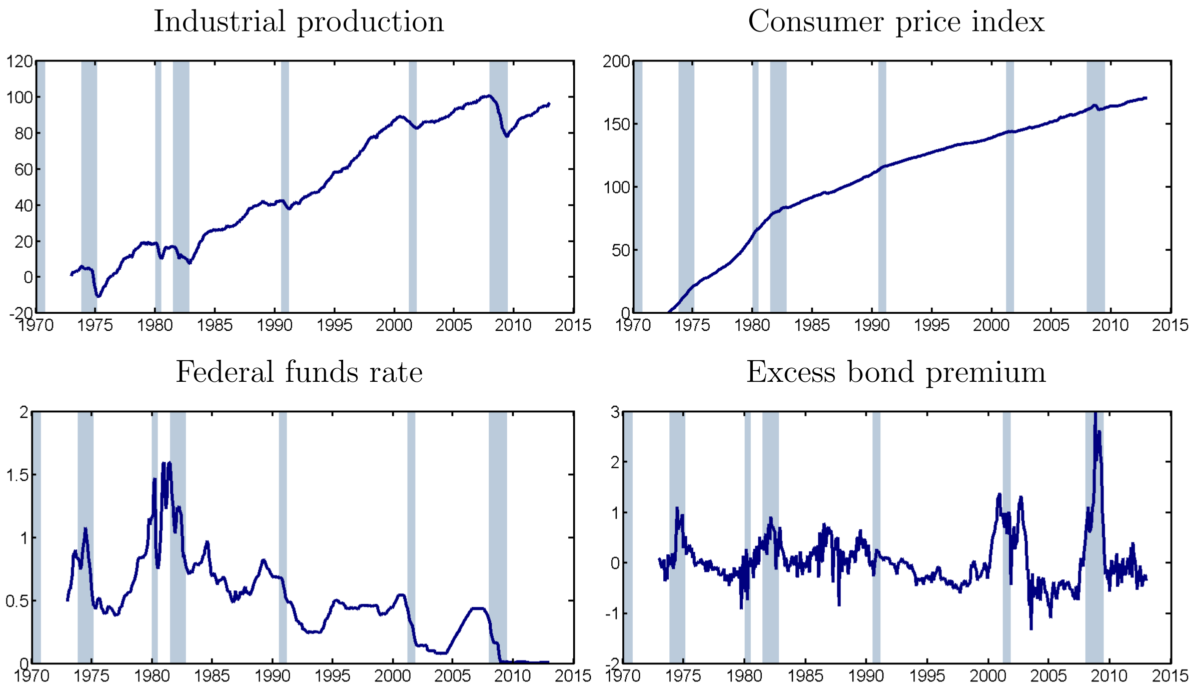

Figure 1 (Clockwise from top) Monthly log industrial production of the manufacturing sector, log consumer price index, federal funds rate, and excess bond premium (bottom) $1973M1-2012M12$. Blue bars indicate NBER recession dates.

the financial channel. Ferreira (2014) isolates the importance of financial uncertainty and finds that it alone can account for 40% of the decline in GDP during the Great Recession.

Data. We use monthly data on log industrial production in the manufacturing sector, log consumer price index, the federal funds rate, and the excess bond premium from $1973M1-2012M12$. We obtained the macroeconomic data from the Federal Reserve Bank of St. Louis FRED and the excess bond premium data from Simon Gilchrist's website. Figure 1 displays the four monthly series.

Model and prior specification. We use a 4-variable specification of our CAIW-in-VAR model. Our preferred function linking the volatility to the mean portion of the model is $f(\Sigma_t) = \log\left(diag(\Sigma_t)\right)$[7]. We choose a VAR(12) for the conditional mean lag length, noting that this lag length essentially removes all autocorrelation from the fitted residuals in a model without stochastic volatility. For the CAIW process, we choose a lag length 1. In estimating the model, we take 100,000 Markov chain Monte Carlo draws from our Metropolis-within-Gibbs sampler. We estimate all parameters in the model. Prior specifications can be found

[7]We also tried $f(\Sigma_t) = diag(chol(\Sigma_t))$. Our results are robust to this specification.

in the appendix. Estimates of the smoothed stochastic volatility process are also in the appendix.

Impulse response functions – restrictions. We consider the real effects of uncertainty shocks with and without a decline in financial sector conditions. To identify the two effects, we propose two different identification strategies. For both identifications, we set $\tilde{\Sigma}_{-1}$ to be the steady state volatility matrix.

Assumption $\mathcal{A}_u$ (Uncertainty shock only) The uncertainty shock, $\overline{v}_t$ satisfy $\mathcal{A}_u$:

$$\mathcal{A}_u : (E[\Sigma_{ii,h}|\sigma_{-1} = E(\sigma), \overline{v}_0^* = e_3; R_t] - E[\Sigma_{ii,h}|\sigma_{-1} = E(\sigma), \overline{v}_0^* = 0; R_t]) > 0 \text{ for } h = 0 \text{ for } i = 1, ..., k.$$

The uncertainty shock only identification specifies that a 1− standard deviation uncertainty shock contemporaneously increases the volatility of all shocks hitting the economy, which is a second moment restriction. This identification of an uncertainty shock is similar in spirit to Jurado et al. (2015), which specifies that an uncertainty shock leads to an increase in the volatilities of many economic series. This assumption imposes that we look at an aggregate uncertainty shock.

Assumption $\mathcal{A}_{uf}$ (Uncertainty shock transmitted through the financial sector) The uncertainty shock, $\overline{v}_t$, satisfy $\mathcal{A}_{uf,1}$ and $\mathcal{A}_{uf,2}$:

$$\mathcal{A}_{u,f,1} : \Big(E[\Sigma_{ii,h}|\tilde{\Sigma}_{-1} = E(\tilde{\Sigma}), \overline{v}_0^* = e_3; R_0] - E[\Sigma_{ii,h}|\tilde{\Sigma}_{-1} = E(\tilde{\Sigma}), \overline{v}_0^* = 0; R_0]\Big) > 0 \text{ for } h = 0 \text{ for } i = 1, ..., k.$$

$$\mathcal{A}_{uf,2} : \Big(E[Y_{f,h}|\tilde{\Sigma}_{-1} = E(\tilde{\Sigma}), \overline{v}_0^* = e_3; R_0] - E[Y_{f,h}|\tilde{\Sigma}_{-1} = E(\tilde{\Sigma}), \overline{v}_0^* = 0; R_0]\Big) > 0 \text{ for } h = 0, ..., H.$$

Relative to identification Assumption $\mathcal{A}_u$, identification Assumption $\mathcal{A}_{uf}$ additionally imposes that financial conditions worsen in expectation for H months. This is a first moment sign restriction. We set $H = 3$, meaning that we restrict financial conditions to worsen for 1 quarter. The motivation for this additional restriction are the empirical results by Caldara et al. (2013) and Ferreira (2014), which emphasize the importance of considering the financial sector when discussing the macroeconomic effects of uncertainty shocks.

Results from Assumption $\mathcal{A}_u$ (Uncertainty shock only). Consider the effects on the macroeconomy of an uncertainty shock (1 standard deviation increase) as shown in figures

2 and 3. Consistent with the sign restriction, the volatilities of all four variables increase in response to the uncertainty shock. The industrial production and excess bond premium volatilities persist the longest, whereas the CPI and federal funds rate volatilities seem to decline more quickly. There is mild evidence of a decline in industrial production, although the response is not significant at the 80% level. The posterior median response is hump-shaped, bottoming out at slightly less -0.2% around 5 months after the uncertainty shock. The price level slightly declines, although the response is at best marginally significant for the initial few months. There is little evidence of any movement in the federal funds rate or financial conditions as captured by the excess bond premium. Overall, these results are consistent with uncertainty shocks having a small effect on the economy.

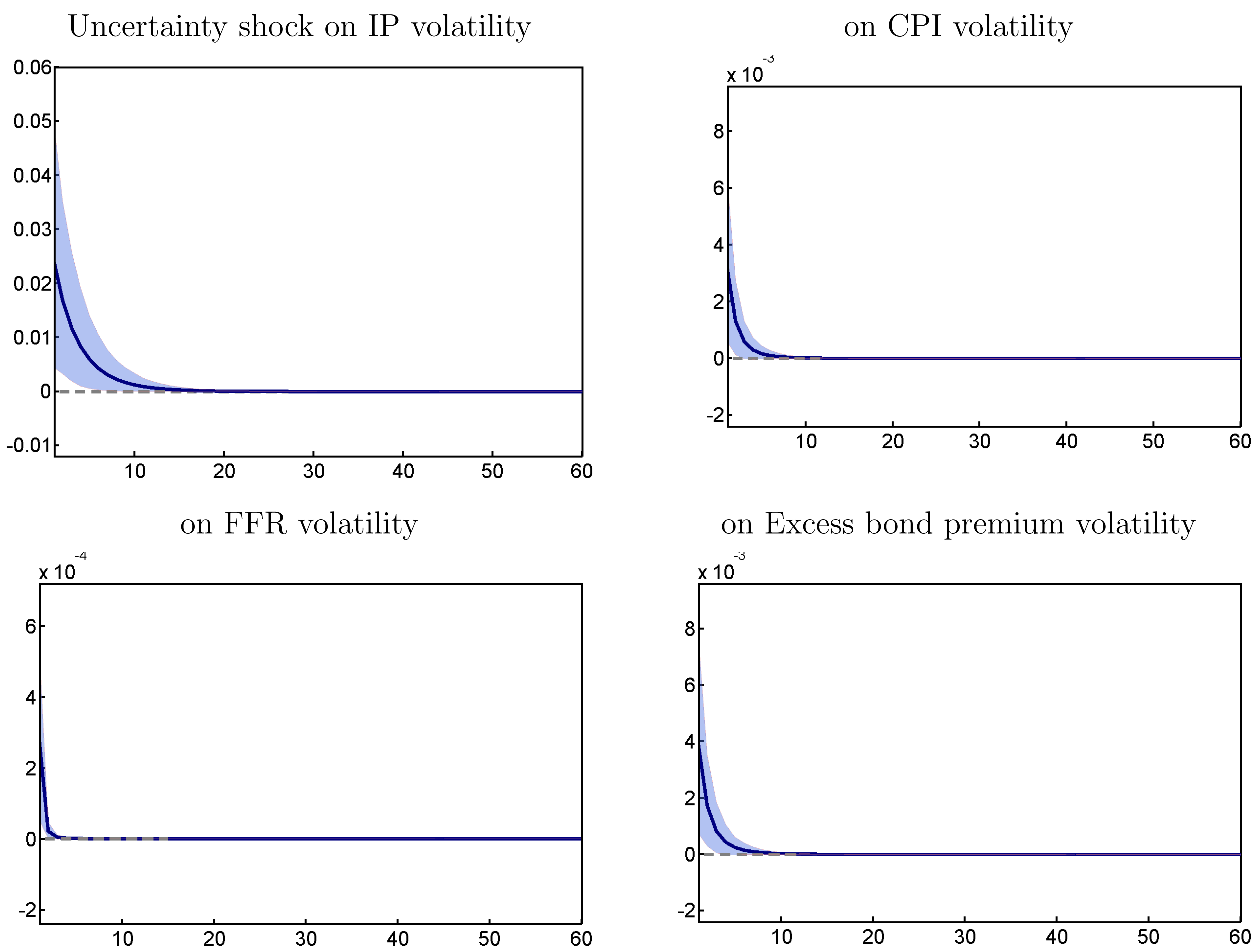

Figure 2 Effect of 1 standard deviation positive uncertainty shock with identification assumption $\mathcal{A}_u$ only. Dark lines give posterior median results. Bands indicate 80% posterior intervals. Time period is in months.

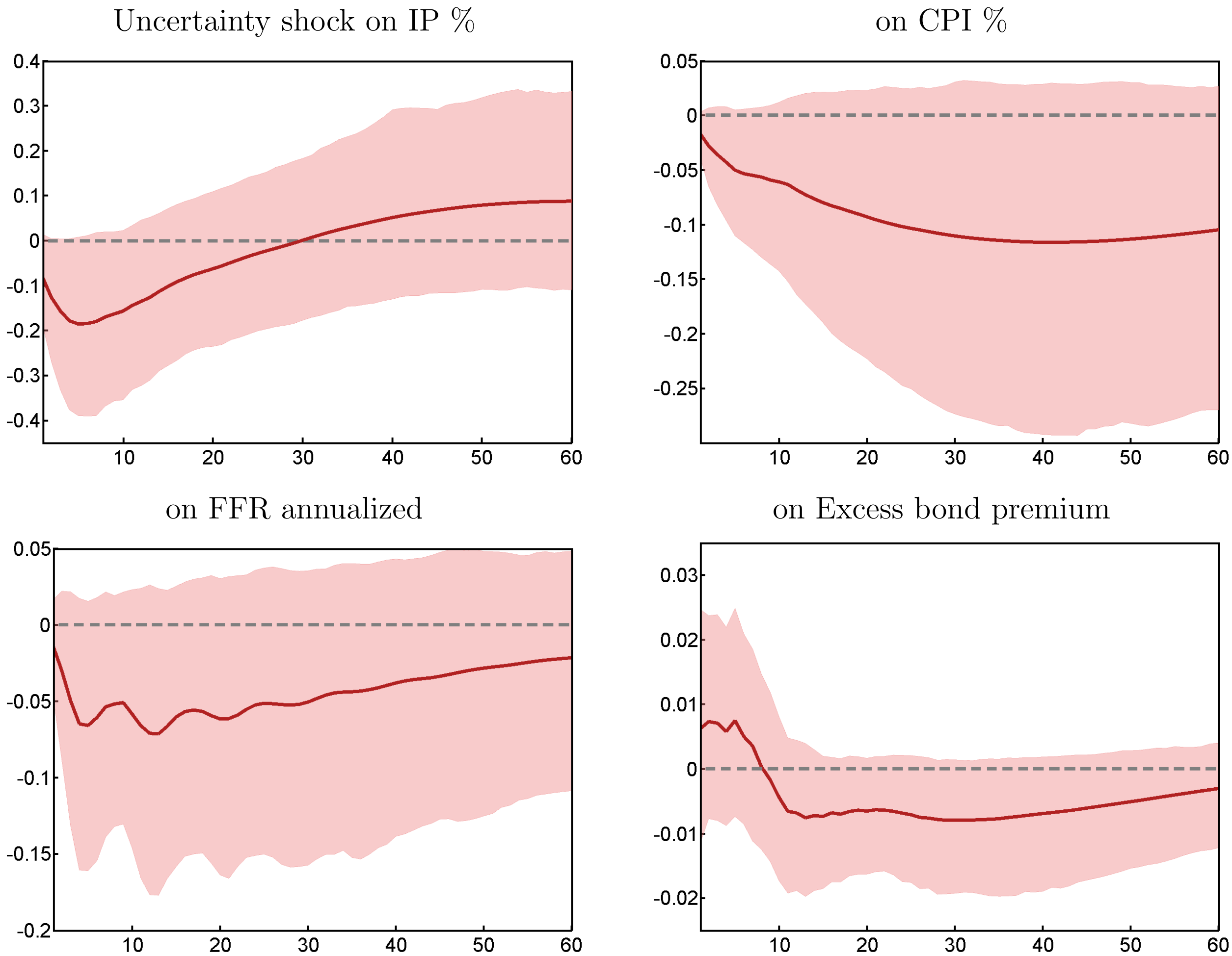

Figure 3 Effect of 1 standard deviation positive uncertainty shock with identification assumption $\mathcal{A}_u$ only. Dark lines give posterior median results. Bands indicate 80% posterior intervals. Time period is in months.

Results from Assumption $\mathcal{A}_{uf}$ (Uncertainty shock transmitted through the financial sector). We impose a further restriction that financial conditions worsen for 3 months following the increase in uncertainty. We interpret this restriction as identifying an increase in uncertainty that leads to a deterioration in financial conditions. This exercise captures an effect similar to the one considered in Caldara et al. (2013). The results are presented in figure 4[8]. Upon restricting the uncertainty shock to move financial conditions, its impact on industrial production becomes significant. Industrial production declines in a hump-shaped and significant manner for 15 months. These results are in line with Bloom

[8]Although in principle the additional sign restriction changes the set of admissable volatility responses, the figures for the volatility responses look quite similar between the two identification schemes. We believe it is more instructive to focus on the first moment responses so we suppress the volatility responses.

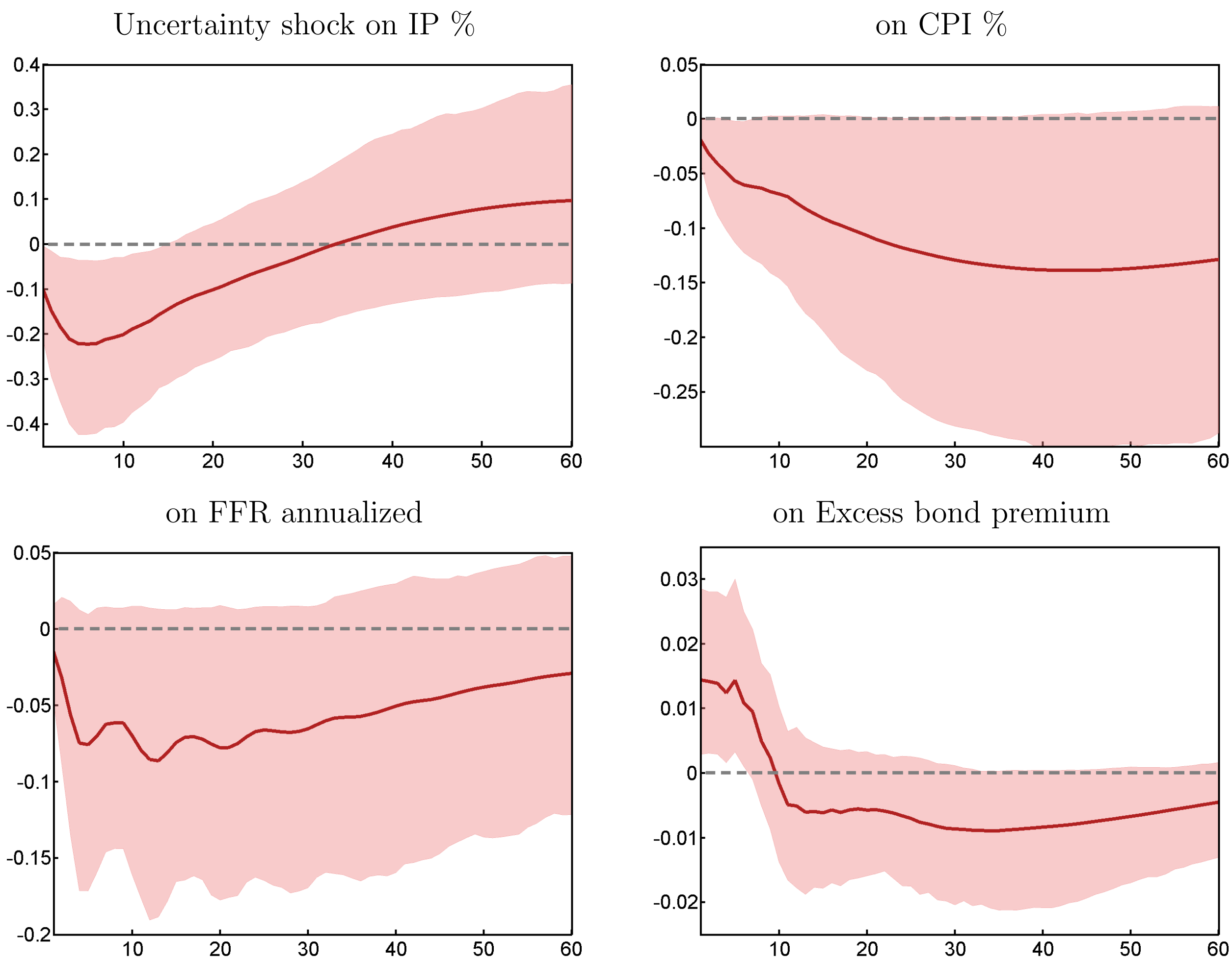

Figure 4 Effect of 1 standard deviation positive uncertainty shock with identification assumption $\mathcal{A}_{uf}$. Dark lines give posterior median results. Bands indicate 80% posterior intervals. Time period is in months.

(2009) in that an uncertainty shock leads to a hump-shaped response in industrial production. Relative to Bloom (2009), the decline in industrial production lasts longer by around 10 months and there is no evidence of a bounceback effect. There is also a stronger response when compared to only imposing assumption $\mathcal{A}_u$ in the price level of around -0.15% that is long-lasting. That uncertainty shocks lead to a decline in the price level is consistent with other studies using alternative identification strategies, such as Leduc and Liu (2012) and Caldara et al. (2013). The federal funds rate does not change by much, perhaps because output and the price level moving the same direction cancel each other out if monetary policy follows the Taylor rule. Financial conditions continue to stay bad for $3-4$ months after the sign restriction. The increase in the excess bond premium is small, with a posterior median of response of around 0.015%. This response is of a similar magnitude to the response of the excess bond premium to an uncertainty shock found in Caldara et al. (2013) when using more macro-based uncertainty proxies (such as forecast disagreement measures and the Economic Policy Uncertainty Index).

5 Conclusion and Future direction

We have advanced the conditional autoregressive inverse Wishart-in-vector autoregression model to evaluate the real effects of uncertainty shocks. We discuss a novel empirical strategy to analyze uncertainty shocks through the first and second moment responses they produce. The strategy allows for the imposition of a limited number of economic restrictions on the uncertainty shocks, thus potentially limiting concerns of misspecification. The paper also presents algorithms to construct impulse response functions to uncertainty shocks. In an empirical application, we evaluate the importance of the financial sector in transmitting uncertainty shocks to the macroeconomy. Our results show that financial conditions are important in transmitting uncertainty shocks to the real economy.

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Appendix

A Details of the posterior sampler

The algorithm runs on the following cycles:

1. $p(\Sigma_t|\ others)$ for $t = 2, ..., T$: : SVs (require a modification)
2. $p(\Sigma_T|\ others)$: SV at the last period (require a modification)
3. $p(\mu, B, \Phi|\ others)$: parameter in the conditional mean equation (NEW)
4. $p(\nu|\ others)$: parameter in Wishart process (same as in original algorithm)
5. $p(C|\ others)$: parameter in Wishart process (same as in original algorithm)
6. $p(A|\ others)$: parameter in Wishart process (same as in original algorithm)

where $p(\theta|others)$ means the conditional distribution of θ given $Y_{1:T}$ and all other parameters except θ. We denote the previous draw as θ^{old}.

Note that the joint posterior distribution is

$$p(\mu, B, \Phi, \nu, C, A, \Sigma_{1:T}|Y_{t:T}) \propto p(Y_{t:T}|\mu, B, \Phi, \Sigma_{2:T})p(\Sigma_{2:T}|\nu, C, A, \Sigma_1)p(\mu, B, \Phi, \nu, C, A, \Sigma_1)$$

where the likelihood function can be decomposed as

$$p(Y_{2:T}|\mu, B, \Phi, \Sigma_{2:T}) = \prod_{t=2}^{T} p(Y_t|Y_{t-1}, \mu, B, \Phi, \Sigma_t).$$

To be able to construct an efficient MCMC sampling algorithm, we break the joint posterior into multiple blocks. For example, the conditional posterior distribution for the multivariate stochastic volatility is decomposed into the following pieces[9],

$$p(\Sigma_{2:T}|\nu, C, A, \Sigma_1, Y_{2:T}) = \left(\prod_{t=2}^{T} p(\Sigma_t|\Sigma_{t-1}, \nu, C, A, Y_t) \right)$$

and iteratively sample Σ_t from $t = 2$ to $t = T$.

Step 1: $\boldsymbol{\Sigma_t}$ for $\boldsymbol{t = 2, ..., T}$ The conditional posterior is

$$\begin{aligned} p(\Sigma_t|others) \propto\ & |\Sigma_t|^{-(\nu+k+1)/2} \exp\left(-\frac{1}{2} tr(S_{t-1}^{-1}\Sigma_t^{-1}) \right) \\ & \times |\Sigma_t|^{-1/2} \exp\left(-\frac{1}{2} tr\left(\epsilon_t' \Sigma_t^{-1} \epsilon_t \right) \right) \\ & \times |S_t|^{-\nu/2} \exp\left(-\frac{1}{2} tr\left(S_t^{-1} \Sigma_{t+1}^{-1} \right) \right) \end{aligned}$$

where we write

$$\epsilon_t = \underbrace{Y_t - \mu - \Phi Y_{t-1}}_{=e_t} - Bf(\Sigma_t) = e_t - Bf(\Sigma_t).$$

Then, we re-write the conditional posterior as

$$\begin{aligned} p(\Sigma_t|others) \propto\ & IW(\Sigma_t|\nu, \widetilde{S}_{t-1}^{-1}) \quad : \text{proposal density} \\ & \times |\Sigma_t|^{-1/2} |S_t|^{-\nu/2} \exp\left(-\frac{1}{2} tr\left(S_t^{-1} \Sigma_{t+1}^{-1} \right) \right) \quad : \text{MH correction 1} \\ & \times \exp\left(-\frac{1}{2} tr\Big(\left[-2 e_t g(\Sigma_t)' + g(\Sigma_t) g(\Sigma_t)' \right] \Sigma_t^{-1} \Big) \right) \quad : \text{MH correction 2} \end{aligned}$$

where

$$g(\Sigma_t) = Bf(\Sigma_t) \quad \text{and} \quad \widetilde{S}_{t-1} = \left(S_{t-1}^{-1} + e_t e_t' \right)^{-1}.$$

[9]The current version of the sampler fixes the initial stochastic volatility at the unconditional mean

$$E[\Sigma_1] = E[\Sigma_t] = (I - \overline{A})^{-1}\overline{C}.$$

A more plausible implementation is to impose the inverse Wishart prior on Σ_1 centered around $E[\Sigma_t]$. Our current implementation can be viewed as a special case of this inverse Wishart prior.

We draw Σ_t based on the independent Metropolis-Hastings algorithm with the inverse Wishart distribution as a proposal distribution, $\Sigma_t^* \sim IW\left(\nu, \widetilde{S}_{t-1}^{-1}\right)$. The acceptance ratio is then,

$$r_{\Sigma_t} = \frac{|\Sigma_t^*|^{-1/2}|S_t^*|^{-\nu/2}\exp\left(-\frac{1}{2}tr\left((S_t^*)\Sigma_{t+1}^{-1}\right)\right)\exp\left(-\frac{1}{2}tr\left([-2e_t g(\Sigma_t^*)' + g(\Sigma_t^*)g(\Sigma_t^*)'](\Sigma_t^*)^{-1}\right)\right)}{|\Sigma_t^{old}|^{-1/2}|S_t^{old}|^{-\nu/2}\exp\left(-\frac{1}{2}tr\left((S_t^{old})\Sigma_{t+1}^{-1}\right)\right)\exp\left(-\frac{1}{2}tr\left(\left[-2e_t g(\Sigma_t^{old})' + g(\Sigma_t^{old})g(\Sigma_t^{old})'\right](\Sigma_t^{old})^{-1}\right)\right)}$$

and we set $\Sigma_t^{new} = \Sigma_t^*$ with probability $max(r_{\Sigma_t}, 1)$, $\Sigma_t^{new} = \Sigma_t^{old}$ otherwise.

We also consider the random-walk-like proposal distribution with the following proposal distribution,

$$\Sigma_t^* \sim IW(\widetilde{w} + k + 1, \widetilde{w}\Sigma_t^{old})$$

where $\widetilde{w}$ is a tuning parameter which governs the variance of the proposal distribution. High $\widetilde{w}$ leads to less variable proposal distribution. As in usual random walk MH algorithm, this proposal distribution is centered on previous draw, Σ_t^{old}. However, the proposal density is not symmetric. The acceptance ratio is then,

$$\begin{aligned}
\widetilde{r}_{\Sigma t} &= \frac{p(\Sigma_t^*|others)/q(\Sigma_t^*|\Sigma_t^{old})}{p(\Sigma_t^{old}|others)/q(\Sigma_t^{old}|\Sigma_t^*)} \\
&= \frac{p(\Sigma_t^*|others)}{p(\Sigma_t^{old}|others)}\left(\frac{|\Sigma_t^*|}{|\Sigma_t^{old}|}\right)^{\frac{(2\widetilde{w}+3k+3)}{2}} \exp\left(-\frac{\widetilde{w}}{2}tr\left(-\Sigma_t^{old}(\Sigma_t^*)^{-1} + \Sigma_t^*(\Sigma_t^{old})^{-1}\right)\right).
\end{aligned}$$

In our application, we set our proposal distribution as a mixture of above two proposal distributions. More specifically, we propose a candidate draw, Σ_t^*:

$$\Sigma_t^* \sim \begin{cases} IW(\nu, \widetilde{S}_{t-1}^{-1}) & \text{with probability } p_\Sigma \\ IW(\widetilde{w} + k + 1, \widetilde{w}\Sigma_t^{old}) & \text{with probability } (1 - p_\Sigma). \end{cases}$$

Step 2: $\boldsymbol{\Sigma_T}$ The full conditional posterior is

$$
\begin{aligned}
p(\Sigma_T|others) &\propto |\Sigma_T|^{-(\nu+k+1)/2} \exp\left(-\frac{1}{2} tr\left(S_{T-1}^{-1}\Sigma_T^{-1}\right)\right) \times |\Sigma_T|^{-1/2} \exp\left(-\frac{1}{2}\epsilon_T' \Sigma_T^{-1}\epsilon_T\right) \\
&\propto |\Sigma_T|^{-(\nu+k+1)/2} \exp\left(-\frac{1}{2} tr\left(S_{T-1}^{-1}\Sigma_T^{-1}\right)\right) \times |\Sigma_T|^{-1/2} \exp\left(-\frac{1}{2}e_T' \Sigma_T^{-1}e_T\right) \\
&\quad \times \exp\left(-\frac{1}{2}\left(\epsilon_T'\Sigma_T^{-1}\epsilon_T - e_T'\Sigma_T^{-1}e_T\right)\right) \\
&\propto IW\left(\Sigma_T^{-1}\middle|\, \nu+1, \left(S_{T-1}^{-1} + e_T e_T'\right)\right) \quad : \text{proposal density} \\
&\quad \times \exp\left(-\frac{1}{2}\left(\epsilon_T'\Sigma_T^{-1}\epsilon_T - e_T'\Sigma_T^{-1}e_T\right)\right) \quad : \text{MH correction}
\end{aligned}
$$

where we define e_T in the same as before. We draw Σ_T based on the independent Metropolis-Hastings algorithm with the inverse Wishart distribution as a proposal distribution, $\Sigma_T^* \sim IW\left(\nu+1, \left(S_{T-1}^{-1} + e_T e_T'\right)\right)$ and therefore the acceptance ratio is

$$
r_{\Sigma_T} = \frac{\exp\left(-\frac{1}{2}\left((\epsilon_T^*)'(\Sigma_T^*)^{-1}\epsilon_T^* - e_T'(\Sigma_T^*)^{-1}e_T\right)\right)}{\exp\left(-\frac{1}{2}\left((\epsilon_T^{old})'(\Sigma_T^{old})^{-1}\epsilon_T^{old} - e_T'(\Sigma_T^{old})^{-1}e_T\right)\right)}
$$

where

$$
\epsilon_T^* = \underbrace{Y_T - \mu - \Phi Y_{T-1}}_{=e_T} - Bf(\Sigma_T^*) = e_T - Bf(\Sigma_T^*).
$$

We set $\Sigma_T^{new} = \Sigma_T^*$ with probability $max(r_{\Sigma_T}, 1)$, $\Sigma_T^{new} = \Sigma_T^{old}$ otherwise.

Step 3: $(\boldsymbol{\mu}, \boldsymbol{B}, \boldsymbol{\Phi})$ First we transform our model into the following multiple regression form,

$$
\widetilde{Y}_t = \widetilde{B}\widetilde{X}_t + \Sigma_t^{1/2}\epsilon_t, \quad \epsilon_t \sim N(0, I)
$$

where p is the number of lags in VAR and

$$
\begin{aligned}
\widetilde{Y}_t &= Y_t' \\
\widetilde{X}_t &= [1, Y_{t-1}', ..., Y_{t-p}', f(\Sigma_t)']' \\
\widetilde{B} &= [\mu, \Phi_1, ..., \Phi_p, B].
\end{aligned}
$$

Then, we can re-write the equation as

$$\Sigma_t^{-1/2}\widetilde{Y}_t = \left(\widetilde{X}_t' \otimes \Sigma_t^{-1/2}\right) vec\left(\widetilde{B}\right) + \epsilon_t,$$

which is a standard multiple regression with homoscedastic errors. The conditional posterior distribution of (μ, B, Φ) is a multivariate normal distribution under the conjugate prior assumption.

Step 4: $\boldsymbol{\nu}$ The conditional posterior distribution of ν is

$$p(\nu|others) \propto \left(\prod_{t=2}^{T} \frac{|S_{t-1}^{-1}|^{\nu/2}}{2^{\nu k/2}\Gamma_k(\nu/2)} |\Sigma_t|^{-(\nu+k+1)/2} exp\left(-1/2tr(S_{t-1}^{-1}\Sigma_t^{-1})\right)\right) \mathbf{1}_{(k+1,M_\nu)}(\nu)$$

where $S_{t-1}^{-1} = (v-k-1)(C + A\Sigma_{t-1}A')$, $\Gamma_k(\cdot)$ is the multivariate gamma function, and $\mathbf{1}_{(k+1,M_\nu)}(\nu)$ is a indicator function takes value 1 if $\nu \in (k+1, M_\nu)$ and 0 otherwise. To draw ν from this conditional posterior distribution, we employ the random-walk Metropolis-Hastings algorithm with a proposal

$$\nu^* = \nu^{old} + e_\nu, \quad e_\nu \sim N(0, \sigma_\nu^2)$$

where the scale of the proposal distribution σ_ν^2 is adaptively chosen so that the resulting acceptance rate is about 30% (Atchadé and Rosenthal, 2005).

Step 5: $\boldsymbol{C}$ The conditional posterior distribution of C is

$$p(C|others) \propto \left(\prod_{t=2}^{T} |S_{t-1}^{-1}|^{\nu/2} |\Sigma_t|^{-(\nu+k+1)/2} exp\left(-1/2tr(S_{t-1}^{-1}\Sigma_t^{-1})\right)\right) \times p_{IW}(C|df, \Psi)$$

where $S_{t-1}^{-1} = (v-k-1)(C + A\Sigma_{t-1}A')$ and p_{IW} is a density function of the inverse Wishart distribution. In this step, we reparametrize C in the following fashion,

$$C = \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ c_{21} & d_{22} & \dots & 0 \\ \vdots & & \ddots & 0 \\ c_{k1} & c_{k2} & c_{k3} & \dots d_{kk} \end{pmatrix} \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ c_{21} & d_{22} & \dots & 0 \\ \vdots & & \ddots & 0 \\ c_{k1} & c_{k2} & c_{k3} & \dots d_{kk} \end{pmatrix}' .$$

This transformation ensures the positive definiteness of C. To draw C from this conditional posterior distribution, we employ the random-walk Metropolis-Hastings algorithm with a proposal

$$c_{ij}^{*} = c_{ij}^{old} + e_{c(i,j)}, \quad e_{c(i,j)} \sim N(0, \sigma_{c(i,j)}^2)$$
$$\log(d_{ii}^{*}) = \log(d_{ii}^{old}) + e_{d(i,i)}, \quad e_{d(i,i)} \sim N(0, \sigma_{d(i,i)}^2)$$

for $(i,j) = \{i = 1,..,k; j = 1,...,k, i \geq j\}$. The scale of the proposal distribution $\sigma_{A(i,j)}^2$ is adaptively chosen so that the resulting acceptance rate is about 30% (Atchadé and Rosenthal, 2005). Note that to compute the acceptance ratio, we need a Jacobian term due to reparametrization,

$$|J| = \underbrace{2^k \prod_{i=1}^{k} d_{ii}^{k+1-i}}_{cholsky\ decomp.} \times \underbrace{\prod_{i=1}^{k} d_{ii}}_{log\ trans.} .$$

Step 6: *A* The conditional posterior distribution of A is

$$p(A|others) \propto \left(\prod_{t=2}^{T} |S_{t-1}^{-1}|^{\nu/2} |\Sigma_t|^{-(\nu+k+1)/2} exp\left(-1/2tr(S_{t-1}^{-1}\Sigma_t^{-1})\right) \right)$$
$$\times p_{TN}(A_{11}|m_{A(1,1)}, V_{A(1,1}, 0, \infty) \prod_{(i,j)\neq(1,1)} p_N(A_{ij}, m_{A(i,j)}), V_{A(i,j)})$$

where $S_{t-1}^{-1} = (v - k - 1)(C + A\Sigma_{t-1}A')$, p_{TN} is a density function of the truncated normal distribution, and p_N is a density function of the normal distribution. Note that the sign of $A_{(1,1)}$ is not identified. Hence, we place the prior distribution over $A_{(1,1)} > 0$.

To draw A from this conditional posterior distribution, we employ the element-wise random-walk Metropolis-Hastings algorithm with a proposal,

$$A_{(i,j)}^{*} = A_{(i,j)}^{old} + w_{i,j}, \quad w_{i,j} \sim N\left(0, \sigma_{A_{(i,j)}}^2\right),$$

where the scale of the proposal distribution $\sigma_{A(i,j)}^2$ is adaptively chosen so that the resulting acceptance rate is about 30% (Atchadé and Rosenthal, 2005) for each (i, j).

B Prior specification

In this section, we present the prior distributions used for the application section. Our benchmark model is the CAIW(1)-in-VAR(12) model with $log(diag(\Sigma_t))$ as a linking function. Parameters in the conditional mean of the model, μ, Φ and B, are assumed to follow independent multivariate normal distributions,

$$\mu \sim N\left(0_4,\ 10^2 \cdot I_4\right), \quad vec(B) \sim N\left(0_4,\ 10^2 \cdot I_4\right), \quad vec(\Phi_1) \sim N\left(vec(I_{16}),\ 10^2 \cdot I_{16}\right)$$
$$vec(\Phi_i) \sim N\left(0_{16},\ 10^2 \cdot I_{16}\right) \quad \text{for } i = 2, ..., 12.$$

where $vec(\cdot)$ is the vectorize operator, $0_\#$ is a $\# \times 1$ vector of zeros, and $I_\#$ is a $\# \times \#$ identity matrix. There are three types of parameters in the volatility equation (A, C, and ν). The parameter A governs the dynamic properties of the volatility matrix process. Each element of A follows an independent normal distribution except the element in the far upper-left corner. The prior distribution for the $(1, 1)$-th element in the A matrix is set to be a truncated normal distribution defined on the positive real line to ensure identification.

$$A(1,1) \sim TN(0.9,\ 0.1^2, 0, \infty), \quad A(i,i) \sim N(0.9,\ 0.1^2) \quad \text{for } i = 2, 3, 4$$
$$A(i,j) \sim N(0, 0.1^2) \quad \text{for } i \neq j.$$

The parameter C determines the long-run mean of the volatility process. We set the prior for it as following an inverse Wishart distribution with scale matrix Ψ and degrees of freedom parameter df. As the Wishart-type distribution is quite a popular prior in the Bayesian literature for a variance covariance matrix, we believe it to be a natural choice for C.

$$C \sim IW\left(4, I_4\right).$$

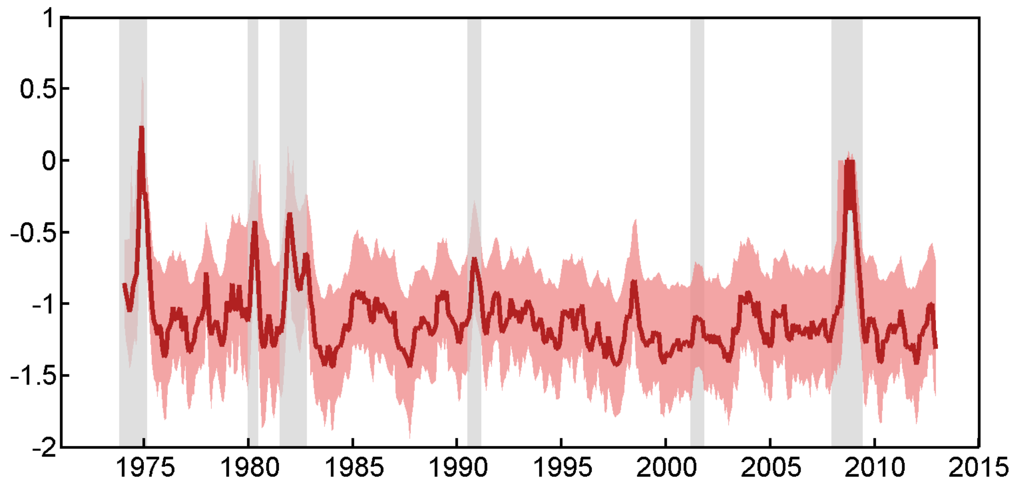
Finally, the prior distribution for ν follows a Gamma distribution with mean 40 and standard deviation 1.

C Posterior estimates

C.1 Stochastic volatility estimates

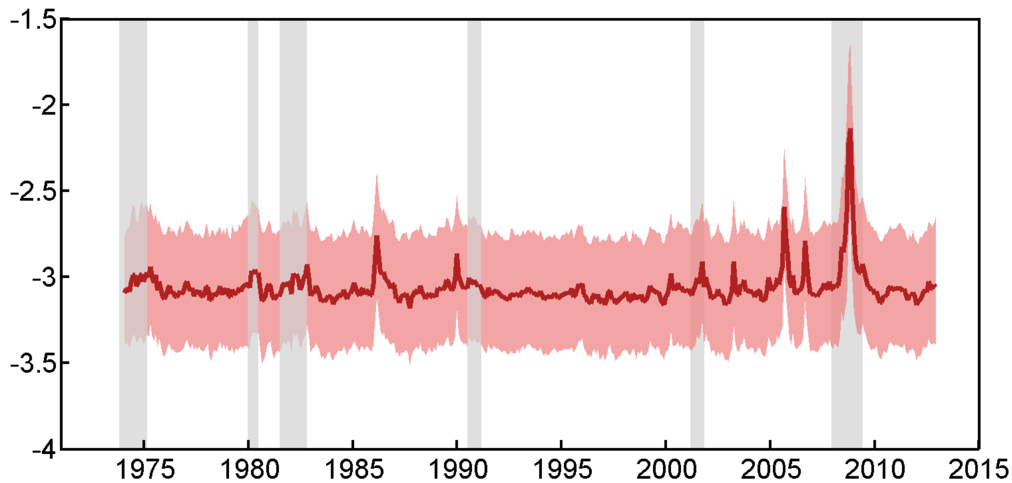
Estimated stochastic volatility series are presented in Figures A-1 through A-4.

Figure A-1 Estimated Stochastic Volatility, IP



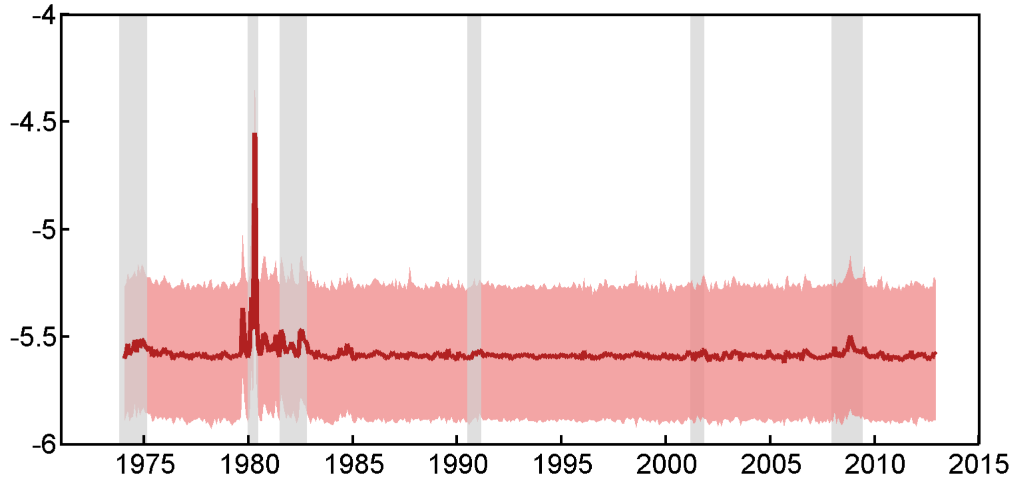
Note: Estimated log stochastic volatility for IP based on the CAIW(1)-in-VAR(12) model.

Figure A-2 Estimated Stochastic Volatility, Inflation



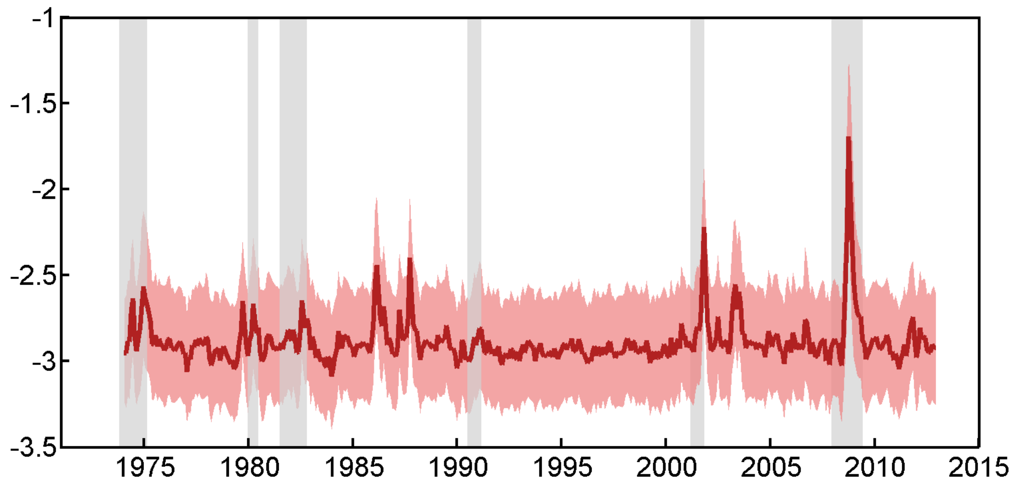
Note: Estimated log stochastic volatility for CPI based on the CAIW(1)-in-VAR(12) model.

Figure A-3 Estimated Stochastic Volatility, Federal funds rate



Note: Estimated log stochastic volatility for the federal funds rate based on the CAIW(1)-in-VAR(12) model.

Figure A-4 Estimated Stochastic Volatility, Excess bond premium



Note: Estimated log stochastic volatility for the excess bond premium based on the CAIW(1)-in-VAR(12) model.