A TALMUDIC BANKRUPTCY SOLUTION: THE CCC PRINCIPLE

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Abstract. Following a bankruptcy, how should we distribute the available assets among the eligible creditors? Most people would accept a proportional distribution—for each claimant, calculate her percentage of the sum of all claims and assign her that same percentage of total assets. However, this is not the only reasonable approach. For example, if every claim is at least as large as total assets, assigning an equal share to every creditor is a sensible solution. Three numerical bankruptcy examples for three claimants, discussed 2,000 years ago in the Talmud, coincide with the above two approaches, but the third case remained a puzzle until recently when modern game theory (Aumann and Maschler, 1985) was enlisted to demystify all cases. This paper explains the unifying principle, the *Contested-Claim Consistency* principle (CCC, or CG-Consistent principle), behind the Talmudic examples. Importantly, it uses different means to better understand the logic behind the CCC bankruptcy allocations and points out the subtle yet important properties behind them. This study aims to clarify the meaning of fairness underlying the CCC allocation, and proposes that CCC may better convey the meaning of the *pari passu* provision that appears in many International Sovereign Debt Instruments.

Keywords: Bankruptcy, CCC principle, Contested Garment, Contested-Garment Consistent, CG-consistent, Contested-Claim Consistent, Distributive Justice, Fair Division, Nucleolus, *pari passu*, sovereign debt, sovereign default

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1. Introduction

It took 2,000 years and two illustrious modern game theorists to crack the following bankruptcy solution from the Babylonian Talmud. Three creditors with claims of 100, 200, and 300 are to split the bankrupt estate. When the estate value is 100, the creditors split the estate evenly at 33 and 1/3 apiece. For an estate worth 200, the creditors get 50, 75, and 75. When the estate value is 300, the creditors receive 50, 100, and 150, in proportion to their claims. These lessons, attributed to Rabbi Nathan, befuddled scholars over the millennia.[2] When the estate is small or large, the even split and the proportional split seem reasonable, considering each case independently. But no one could explain the bankruptcy division for an estate worth 200. Putting all three cases together is even more mysterious; the underlying principles dictating the numerical examples were thus hidden in an abyss for ages.

In the mid-nineteen-eighties, Robert Aumann and Michael Maschler unexpectedly discovered that the Nathan examples prescribe the same solutions as those from the nucleoli of the corresponding coalitional games.[3] Not being armed with modern game theoretical concepts, it is inconceivable that the wise sage could develop the bankruptcy solutions through the same means. This inspired the authors to find alternative mechanisms by which Rabbi Nathan came to his numerical lessons. Aided by research on nucleolus, Aumann and Maschler discovered that the Nathan solutions can be explained through the consistent application of another Talmudic principle – the Contested Garment principle.[4] Their 1985 *Journal of Economic Theory* article presents a precise definition for a non-game-theoretic bankruptcy solution, which generalizes the Nathan numerical examples. They call it the Contested-Garment Consistent principle, or the CG-consistent principle.[5,6] The term "garment" in the name of the principle is equivalent to *claims* in the bankruptcy problem. To respect the historical source while giving it a modern flavor, this article interchanges contested garment with contested claim. For ease of discussion, the Contested-Garment Consistent principle will be renamed the CCC principle, an abbreviation for the *Contested-Claim Consistent* principle.

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[2] The Nathan examples are presented in the Tractate of *Kethuboth* 93a (chap.10, mishnah 4).

[3] A coalitional game centers its attention on coalitions formed by subsets of players in a game. The nucleolus further focuses on dissatisfaction for any coalition; it is the solution that minimizes the largest dissatisfaction among all possible coalitions (Hill 2000). The nucleolus of a coalitional game exists and is unique, and it is group- as well as individually-rational.

[4] The Contested Garment example is presented in the Tractate of *Bava Metzia* (2a) of the Talmud. This example explains how a piece of contested garment should be divided between two creditors, hence the term "garment."

[5] Aumann and Maschler (1985) actually presented "three different justifications of the solution to the bankruptcy problem that the nucleolus prescribes…" However, most attention is paid to the bankruptcy solution derived by applying the CG-consistent principle.

[6] The CG-consistent principle requires the consistent application of the Contested-Garment principle for *any pair of creditors*.

Aumann and Maschler first introduce the simple idea of equal sharing of contested claim for the case of two creditors. They then extend the principle of equal sharing of *c*ontested *c*laim by incorporating the feature of *c*onsistency to establish the general *CCC* principle for dividing a bankrupt estate. They show that the CCC principle gives rise to a unique solution for any bankruptcy problem; the unique division matches each of the solutions in Nathan's three numerical examples, and it also coincides with the nucleolus of the properly defined coalitional games. The mathematical, non-game-theoretical presentation of the CCC principle appears straightforward, but the general solution to any bankruptcy scenario is not easy to find and the subtle properties of the bankruptcy divisions are often elusive. Not surprisingly, the little understood CCC principle was not widely applied or studied. Quite apart from the importance of understanding this age-old bankruptcy solution, it is hoped that a better understanding will make the CCC principle a strong alternative to the ubiquitous proportional principle in some bankruptcy cases. In particular, it will be exciting to see how a deeper understanding of an age old allocation problem can shed light on recent sovereign bankruptcy cases. For over 100 years, the international debt instrument incorporated the *pari passu* clause, even though its exact meaning was not clear.[7] Recent court cases involving Peru, Argentina, and other countries accepted the interpretation of *pari passu* as a proportional allocation.[8] Elsewhere, we will argue that the CCC principle may better fit the underlying idea of *pari passu*.[9]

This article pushes the envelope in many different directions: to supplement the pioneering work and the path-breaking insights of Aumann and Maschler (1985); to further our understanding of the Talmudic bankruptcy solution; and to search for the basis upon which the current international sovereign debt crisis may be reasonably connected to the CCC principle. Section 2 reiterates the important discovery by Aumann and Maschler (1985) of the CCC principle, employing the non-game theoretic approach. Section 3 presents an alternative definition, one that was briefly mentioned but not well-developed in Aumann and Maschler (1985). The equivalence of the two approaches is shown. Section 4 discusses the difference between (1) nominal gains and nominal losses across creditors, to which the literature pays attention, and (2) gains and losses under appropriate bounds, as stressed in this article. The alternative way of

[7] The Latin phrase *pari passu* means "in equal step."

[8] For example, see *Elliott Assocs., L.P.*, General Docket No. 2000/QR/92 (Court of Appeals of Brussels, 8th Chamber, 26th Sept. 2000) and the many court cases concerning Argentina such as *NML Capital v. Republic of Argentina*, 621 F.3d 230 (2d Cir. 2010). On November 21, 2012, District Judge Griesa issued orders resolving the issues remanded to him by the Second Circuit in the Circuit Court's decision of October 26 and gave a huge victory to the holdout plaintiffs against Argentina. The saga of legal maneuvers continued and finally took a different turn at the end of February 2016, after newly elected President Mauricio Macri reversed the earlier administration's stance, the negotiation with the holdout funds now had the blessings of Judge Griesa. It appears that "the trial of the century" may soon be over.

[9] Investigating whether the principle better fits the meaning of *pari passu* than the alternative proportional allocation inspired this paper's study of the CCC principle. Two other papers and the current one form a trio to investigate this question. Fon (2016*a*) compares the CCC and the proportional principles. Fon (2016*b*) argues in detail why the CCC principle should/could be embedded in the *pari passu* clause in international sovereign debt instruments.

viewing gains and losses is in fact more revealing; it provides a new understanding of the criterion behind the CCC divisions.

The remainder of the article concerns the pattern of CCC bankruptcy allocations. Although an implementation for finding CCC allocations is known and somewhat easy to understand, especially for cases with few creditors, a concise table of divisions for any combination of claims and estate values has not appeared in the literature.[10] Section 5 establishes a table showing CCC shares assigned to each creditor for the building-block case of two creditors with any claim and estate values, highlighting the boundary scenarios. Section 6 proves and extends the CCC division for the case of three creditors. The table is then applied to the historical case discussed by Rabbi Nathan, in which the claims owed to the creditors are 100, 200, and 300. The table shows the assignment of the CCC divisions for *any* value of the estate; providing a complete answer to the millennia-old puzzle and extending other numerical tables in the literature.

Section 7 concludes the investigation by explaining the different aspects of the CCC bankruptcy allocations, and then briefly explores the potential application of the CCC divisions to the current *pari passu* clause commonly employed in international sovereign debt instruments. The interest in the meaning of *pari passu* began in the year 2000 with a case concerning Peruvian debt issues in the Belgium court, and continues to play an important role in the Argentine debt default cases battled in New York and British courts.[11,12] Considering that many outstanding sovereign instruments with the *pari passu* provision are around and that many countries are at risk financially, it is time to investigate exactly what *pari passu* should or could mean and whether it makes sense to apply the CCC principle to the provision.[13] The paper ends with some thoughts on this issue. But first, we must fully understand what is behind the simple yet mysterious idea of CCC.

2. The CCC Principle and the Equal Division of Contested Claim

We start our investigation by reiterating the Talmudic bankruptcy lesson attributed to Rabbi Nathan. The numerical recommendations are presented in Table 1, where rows present each claimant's share of the bankruptcy allocation, and columns represent different values of the

[10] In a different direction Kaminski (2000) proposes a physical solution to the problem using a hydraulic analogy; this approach can be visualized with the help of graphs.

[11] See *Knighthead Master Fund LP v. The Bank of New York Mellon* (2014), HC-2014-000704 in the English High Court of Justice, Chancery Division, in London, England.

[12] Possibly inspired by their country's inadvertent involvement in the *pari passu* saga, in June 2015 the Finance Committee of the Belgian Parliament passed a provision to attempt to limit the amount that hedge funds can collect on debts. Aggressive hedge funds pay a deep discount for a distressed outstanding debt instrument but demand full payment plus interest from the debtor. The *Elliott* case in 2000 was the first successful attempt applying a legal strategy, *supra* note 8.

[13] For example, debt instruments of Ukraine and Cyprus contain the same *pari passu* clause.

bankrupt estate. The common lesson embedded in these three numerical examples is murky: it is not clear how the allocation scheme evolves from an even-split with a small estate value to a proportional division when the estate increases in value. Exactly what rule dictates the division for an estate worth 200? Further, what happens when the estate increases beyond 300?

Claim \ Estate	100	200	300
100	33 1/3	50	50
200	33 1/3	75	100
300	33 1/3	75	150

Table 1. Rabbi Nathan's Recommendation for the Bankruptcy Problem

The mysterious lessons deterred application of this principle for two millennia. In the 1980s, Aumann and Maschler translated the three bankruptcy examples into game-theoretic models, tested them against all known solutions, and discovered that only the nucleolus, a solution concept invented by David Schmeidler (1969), generates exactly the same divisions as the Nathan examples. Aumann and Maschler (1985) present three non-game theoretic justifications to identify the bankruptcy solution. We follow the subsequent literature by examining their first and most approachable characterization.[14] Aumann and Maschler found that two principles are behind the hidden lessons. First, a basic principle allocates a deficit sum between two creditors, and second, the allocation principle is consistently applied to any pair of creditors. While the consistency requirement is not at all transparent from Nathan's numerical examples, the 2-creditor building block case advocated by Aumann and Maschler resonates another easily understood Talmudic lesson, the Contested Garment (Contested Claim) principle. According to this principle, "Two hold a garment; one claims it all, the other claims half. Then the one is awarded three-fourths, the other one-fourth."[15] Since the lesser claimant only claims half of the garment/estate, she concedes half to the greater claimant.[16] As the contested claim only involves half of the estate, the two claimants equally share this, leaving the greater claimant with three-quarters (one half plus one quarter) and the lesser claimant with one-quarter.[17] Thus, the "Contested Claim principle" always implies equal sharing between the creditors.

[14] For example, see Aumann (2002); Elishakoff (2011), and Schecter (2012).

[15] The Tractate of Bava Metzia (2a) in the Talmud starts off with this contested garment example.

[16] We use "she" to refer to creditors. This is done in deference to the Talmudic case law concerning bankruptcy which inspires the CCC principle; creditors were women in the classic examples.

[17] Aumann (2002) attributes this to Rashi, who "explains here that the claimant to half the garment 'concedes…that half belongs to the other, so that the dispute revolves solely around the other half. Consequently, … each of them receives half of the disputed amount.' "

While the contested claim principle is easy to understand, the reason why Nathan's unifying lesson eluded investigative attacks through the Ages lies in the *consistent* application of the Contested Claim principle for any pair of creditors. We combine the two requirements as the Contested-Claim Consistent (CCC) principle. Consistency means that any two creditors always divide the total amount assigned to them by applying the Contested Claim principle. Taking the sum of the amounts given to *any two* creditors as "the value of an estate," the Contested Claim principle divides this sum between these two creditors. Hence, the CCC division of a bankrupt estate among *any* number of creditors is such that any two of them always divide the sum they jointly receive according to the principle of Equal Sharing of Contested Claim.

Although it is extremely difficult to detect from the Talmudic numerical examples, the consistency requirement is easy to grasp and confirm once the bankruptcy division is proposed. To see clearly that the CCC principle is applied throughout Nathan's examples, first examine the allocation when the estate is 100. Any two creditors are jointly awarded 66.66. Since either creditor's claim exceeds this amount, nothing is conceded and the entire 66.66 is contested. Sharing this contested claim equally, both creditors receive 33.33. Next, for an estate worth 200, take the simple case between the 200- and the 300-claimants. They receive the joint sum of 150, which falls below both claims. So the entire 150 is contested and split equally, with 75 awarded to each claimant. Now consider what happens between the 100-claimant and either of the remaining claimants. In each case the two claimants are jointly assigned 125, where the sum exceeds the claim of the lesser 100-claimant but not the greater claimant. The 100-claimant concedes 25 to the other claimant and contests 100 of the estate, while the other claimant concedes nothing to the 100-claimant and contests the whole estate. This means that the contested claim of 100 is equally divided between the two claimants, giving 50 to the 100-claimant and 25+50 to the greater claimant. Lastly, for an estate value of 300, it is straightforward to confirm that the shares awarded to each pair of claimants follow the CCC principle, where the combined awards for any pair of claimants always falls between the two claims.

Once a CCC allocation is known, confirming that the CCC principle is consistently applied is straightforward. On the other hand, even after pointing out that the CCC principle should be applied, finding the CCC allocation can be daunting, especially with many creditors. We return to this practical issue later. Fundamentally, once the mystery of consistency is resolved, understanding the underlying characteristics of the CCC allocations lies solely in the basic case of equal division of a contested claim by any 2 creditors. Thus, to fully comprehend the CCC principle, we need to carefully study the building-block case in which two creditors share a bankrupt estate.

Mathematically, Aumann and Maschler (1985) describe a 2-creditor problem with estate E and claims d_1 and d_2 (with $d_1 \leq d_2$). They first assign the concession offered to each claimant. The share (s_i) awarded to each creditor *i* is the sum of the conceded amount and half of the residual of the two concessions.[18] Formally, let $c_i^{\sim} = \max\{E–d_j,0\}$ be the amount conceded by creditor *j* in favor of creditor *i*. $c_i^{\sim}$ will be referred to as the *de facto* concession awarded to *i* (by the other claimant *j*). Intuitively, a positive amount is conceded to *i* only if creditor *j*'s claim is less than the estate value *E*; the *de facto* concession is zero if *j*'s claim is greater than the estate value because concessions cannot be negative. More specifically, when $E>d_j$, the *de fa*cto concession $c_i^{\sim} = E–d_j$ is positive and matches our intuition; when $E<d_j$, the *de fa*cto concession $c_i^{\sim}$ is nil. Given the *de facto* concessions, the disputed amount, or the contested claim, to be shared equally is $E–c_1^{\sim}–c_2^{\sim}$, and the share given to creditor *i* under the equal division of the contested claim principle is $s_i = c_i^{\sim}+(E–c_1^{\sim}–c_2^{\sim})/2$.[19]

When Aumann and Maschler abstract from the Talmudic lesson of equal sharing of contested claim and transpose the underlying principle to mathematical expressions, they incorporate two subtle extensions. The first is that any amount claimed beyond the estate value by a creditor is irrelevant; only the fact that the claim is at least as large as the estate is important.[20] This is incorporated in the definition of the *de facto* concession $c_i^{\sim} = \max\{E–d_j,0\}$, in which whenever a claim d_j is greater than the estate *E*, the negative difference $E–d_j$ must be replaced by 0. Equivalently, any claim exceeding the worth of the estate, no matter how large, is treated as if the claim equals the size of the estate, because it is pointless to dispute any claim amount beyond the value of the available estate. This makes sense even though the historic Talmudic garment example does not showcase the case in which a claim exceeds the entire garment/estate; the focus there is on the sharing aspect between the creditors.

The second subtle extension established in Aumann and Maschler (1985) lies in the concept and naming of contested claim. In the original contested garment example, one creditor claims the whole garment while the other claims half, making the idea of taking the contested claim to be half of the garment pretty easy to accept. However, a little reflection suggests that the concept of contested claim may not be unique:[21] whenever each individual claim is less than the estate, both claims can be considered contested. For example, if the creditors' claims are 1/2 and 2/3 of the

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[18] Moulin (2003, p.38) refers to this as the concession property.

[19] For ease of extending the Aumann and Maschler (1985) mathematical definition for contested claim allocations, we introduce a slightly different notation $c_i^{\sim}$ to denote concessions, and highlight the restriction that concessions cannot be negative by calling them *de facto* concessions. Aumann and Maschler call the amount allocated to a creditor an "award;" we mostly call the same concept a "share." This is done because award gives a positive connotation, while the forthcoming alternative definition provided in this article concentrates on loss suffered, which conveys a negative connotation. Note also that the term share represents the amount, not the proportion of the bankrupt estate given to a creditor.

[20] Aumann and Maschler state this fact clearly at the beginning of their article: "Any amount of debt to one person that goes beyond the entire estate might well be considered irrelevant; you cannot get more than there is."

[21] In Nathan's examples, dividing what a pair of creditors receives in accordance with the CCC principle involves either no concession or one concession: the concept of contested claim is straightforward.

estate, one can argue that both 1/2 and 2/3 are good candidates for contested claim (by one creditor only).[22] When a unique and appropriate concept for contested claim by both creditors is lacking, the crucial concept involved is actually not a contested sum. In fact, as Aumann and Maschler point out, the crucial concept is the residual of the two (*de facto*) concessions from the estate. This residual is what should be divided equally and be added to the *de facto* concession offered by the opponent creditor. Thus, it is important to recognize that the term "contested claim" is short-hand for the residual of *de facto* concessions from the estate.

Aumann and Maschler did not take the residual-definition rabbit out of a hat. Their precise definition of contested claim $E–c_1^{\sim}–c_2^{\sim}$ as the residual of the *de facto* concessions has the backing of another Talmudic source,[23] which was cited in Aumann (2002). A grandfather dies and is survived by three grandsons. One troublesome grandson claims 1/2 of his grandfather's estate[24] while a coalition of two other grandsons jointly claims 2/3 of the grandfather's inheritance.[25] Thus, 1/2 of the estate is conceded by the troublesome grandson to the coalition of the other grandsons, and 1/3 of the estate is conceded by the coalition to the troublesome grandson. This leaves 1/6 of the estate to be split equally between the troublesome grandson and the coalition, and the troublesome grandson ends up with 5/12 (1/3+1/12) while the coalition of the other grandsons receives 7/12 (1/2+1/12) of the estate. The theoretical definition given by Aumann and Maschler describes exactly this division of the residual of the concessions from the estate.

The two extensions made in Aumann and Maschler (1985) are reasonable and retain the original important spirit of equal division of the part of the estate that both creditors considered hers. The shares given to the two creditors under the principle of equal sharing of contested claim are as follows:

$$s_1^G = c_1^{\sim}+(E–c_2^{\sim}–c_1^{\sim})/2 = (E+c_1^{\sim}–c_2^{\sim})/2, \text{ and}$$
$$s_2^G = c_2^{\sim}+(E–c_2^{\sim}–c_1^{\sim})/2 = (E–c_1^{\sim}+c_2^{\sim})/2.$$

The superscript *G*, as in Gain, is inserted here to highlight the gain-sharing approach of the CC divisions advocated by Aumann and Maschler. The explicit solutions of the creditors' shares, as

[22] One might surmise that $\min\{d_1,d_2\}$is a good candidate for contested claim, which seems compatible with the Talmudic garment example. Let us take contested claim to equal $\min\{d_1,d_2\}$. Suppose the estate is larger than both claims: $d_1<d_2<E$. Concessions then equal $E– d_2$ and $E– d_1$, and half of the contested claim is designated as $d_1/2$. The shares allocated to the two creditors then total $[E– d_2+ d_1/2]+[E– d_1+ d_1/2] = 2E– d_2$. But in general this sum does not equal the estate E, which should be the case. We conclude that in general, contested claim cannot be defined as the minimum of the two claims.

[23] This appears in the Tractate of *Bava Metzia* in the Tosefta (a secondary source contemporaneous with the Mishna).

[24] This grandson claims that he is fathered by the older son of his grandfather.

[25] These two other grandsons claim that all three grandsons are fathered by the younger son of their grandfather. In Jewish tradition, if a brother dies without an heir, a surviving brother takes the widow as a wife in order to provide an heir to the dead brother. The paternity of a child (the troublesome grandson in this example) can thus be in dispute if the widow gives birth eight or nine months after the death.

functions of the estate E and the creditors' claims d_1 and d_2, will be presented in the next section, after providing an alternative definition of the CC allocation.

Before that, it is worth reiterating that Aumann and Maschler (1985) presents three different non-game theoretic justifications for the bankruptcy solution. They begin by discussing the CC division for any two creditors, which is the building block of the final allocation for all creditors. Their first justification (consistency) extends application of the 2-creditor CC division to *any* number of creditors; we highlight this approach by calling it the CCC principle. Their second justification (self-duality) stresses the fact that the single CCC rule, which assigns the CCC division to each bankruptcy problem, treats gains and losses in the same way, and the qualitative changes from gains to losses occur at half of total claims. More precisely, in these two approaches, Aumann and Maschler (1985) highlights the qualitative change in viewing bankruptcy divisions at the point where the estate equals half of the total debt. Before the half way point the divisions are viewed as gains, and after the half-way point they are regarded as losses. This symmetric property with respect to half of total debt has Talmudic backing. However, this article will show that in one respect the half-way point is not important, and in another respect, a different mid-point plays a more important role behind the CCC division solution (depending on the combined awards given to the two creditors in question). In fact, the third justification (coalition formation) in Aumann and Maschler (1985) does not enlist the use of the qualitative change at half of the total debt.

After presenting the justifications without the use of modern game theory, Aumann and Maschler (1985) proves that the proposed solution is the nucleolus of an appropriately defined cooperative game. A corresponding coalitional game to the bankruptcy problem can be formed by considering a group of N creditors. The worth v of any coalition consisting of a subset S of creditors is the total payoff that the coalition can obtain by itself without the help of other creditors. In the bankruptcy problem, the worth of the coalition is naturally taken to be zero or what is left of the estate E after each member outside of the coalition receives her complete claim. In particular, in the case of two creditors, the worth of creditor 1 is what she can assure herself $c_1^{\sim}$, and creditor 2's worth is $c_2^{\sim}$. Nucleolus, a single-valued solution to the game, ensures stable coalitions by identifying an allocation that minimizes complaints. It generalizes the standard solution of a 2-person game, under which each creditor is given the amount that she can assure herself and they divide the remainder equally between them. Specifically, with two creditors, if $s_1–c_1^{\sim} > s_2–c_2^{\sim}$ for example, the excess beyond her worth assigned to creditor 1 is larger than the excess given to creditor 2, and creditor 2 will complain. Minimizing complaints across creditors then requires $s_1 –c_1^{\sim} = s_2–c_2^{\sim}$. Together with the requirement that the two creditors share what is given to the two of them, $s_1+s_2 = E$, the two conditions lead to s_1^G and s_2^G given above, exactly what the CC division prescribes. To recap, nucleolus generalizes the standard solution in cooperative game theory and CCC is the consistent application of the CC division which matches the standard solution exactly.

With this brief survey, we are now ready to discuss an alternative definition of the CC division, the building block of the proposed CCC bankruptcy solution.

3. The CCC Principle and the Equal Division of Joint Loss

As suggested in Aumann and Maschler (1985, p.204), the creditors may well consider the bankruptcy allotment a loss rather than an award. This suggests that the bankruptcy rule should make no difference whether the outcome is considered an award or a loss. To proceed, we provide an alternative definition of the CC division to that discussed in the last section. Previously, the CC allocation specifies an equal division of the possible recovery from the bankrupt estate for any pair of creditors, the residual of the concessions from the estate. Alternatively, instead of studying how two creditors equally share the excess of the bankrupt estate beyond the conceded amounts, what happens if the creditors equally share the loss, the shortage of the estate from the total claim?[26] Will the same allocation indeed be obtained? To focus on loss, it is important to recognize that the relevant concept here is not the shortages of the nominal (the value of) claims d_i from the estate, but the shortages of the *de facto* claims, with appropriate bounds, from the estate. Much akin to the need to bound each creditor's *de facto* concession from below, we need to bound each creditor's claim from above. Previously, an individual *de facto* concession to creditor i is defined as $c_i^{\sim} = \max\{E–d_j,0\}$; presently, an individual *de facto* claim from creditor i is defined as $d_i^{\sim} = \min\{d_i, E\}$.[27] Specifically, when d_i exceeds E, it is pointless for creditor i to claim anything more than E, so the *de facto* claim for creditor i is truncated and capped at the value of the estate: $d_i^{\sim} = E$. In the special case when i's claim d_i is less than the estate value E, the creditor cannot ask for anything beyond what the estate owes her, so the nominal claim becomes the *de facto* claim: $d_i^{\sim} = d_i$.

Given the *de facto* claims of creditors 1 and 2, the *de facto* joint loss under bankruptcy to be borne by the pair is now $d_1^{\sim}+d_2^{\sim}–E$. Similar to their prominent derivation of a CC allocation, where a creditor's individual gain is added to the *de facto* concession, Aumann and Maschler also implied that the CC allocation can be obtained by equally dividing the *de facto* joint loss

[26] In the beginning of their article, when they explain the principle behind the classic contested garment example in which one claims all and the other claims half, Aumann and Maschler (1985, Footnote 6) point out that "Alternatively, … the loss is shared equally." As we shall see, the idea that loss, without any qualifications, is equally shared is correct only if each creditor's claim is no greater than the estate. This requirement holds in the contested garment example.

[27] As noted in the last section, the fact that whenever her claim exceeds the estate, a creditor should adjust her claim to the total estate available was incorporated in the Aumann and Maschler definition of the *de facto* concession. Although no discussion is offered, the concept of *de facto* claim is mentioned in Footnote 8 of Aumann and Maschler (1985). Moulin (2003, p.37) calls the *de facto* claim definition a truncation.

from the creditors' claims.[28] Here, the individual loss should be subtracted from her *de facto* claim, not from her nominal claim. Again, the *de facto* concept for the claims must be incorporated because claiming anything that is not there has no practical consequence. Thus, under the principle of equal sharing of *de facto* joint loss, the shares (with a superscript L as in Loss) of the bankrupt estate allotted to the creditors, presented as the *de facto* net loss of each creditor, are given below:

$s_1^L = d_1^{\sim} – (d_1^{\sim} + d_2^{\sim} – E)/2$ and
$s_2^L = d_2^{\sim} – (d_1^{\sim} + d_2^{\sim} – E)/2$.

Note the symmetries between the definitions for the shares given to each creditor under the equal sharing of the *de facto* contested claim s_i^G in the last section, and under the equal sharing of *de facto* total loss s_i^L in this section. Since the relative magnitude of an individual claim against the entire estate heavily influences the individual *de facto* concession and *de facto* claim, the shares assigned to the creditors ought to depend crucially on the relations among the estate and the claims as well. Thus, to confirm that the alternative definition assigns the same allocations to the creditors as the Aumann and Maschler (1985) definition, Tables 2 and 3 in the following are classified according to the different ranges of E.

	$c_1^{\sim}$	$c_2^{\sim}$	$E – c_1^{\sim} – c_2^{\sim}$	$s_1^G = c_1^{\sim} + (E – c_1^{\sim} – c_2^{\sim})/2$	$s_2^G = c_2^{\sim} + (E – c_1^{\sim} – c_2^{\sim})/2$
$E \le d_1 \le d_2$	0	0	E	$E/2$	$E/2$
$d_1 \le E \le d_2$	0	$E – d_1$	d_1	$d_1/2$	$E – d_1/2$
$d_1 \le d_2 \le E$	$E – d_2$	$E – d_1$	$d_1 + d_2 – E$	$(E + d_1 – d_2)/2$	$(E + d_2 – d_1)/2$

Note: $c_1^{\sim} = \max\{E – d_2, 0\}$; $c_2^{\sim} = \max\{E – d_1, 0\}$.

Table 2. 2-Creditor Bankruptcy Allocation (s_1^G, s_2^G) under Equal Sharing of Contested Claim (Aumann and Maschler's definition)

[28] Aumann and Maschler (1985, Footnote 8) state, "Alternatively, one may argue that neither claimant i can ask for more than $\min(E, d_i)$. If each claimant is awarded *this amount*, the total payment may exceed the estate; the excess is deducted in equal shares from the claimants' *awards*. This procedure leads to the same payoff …" (Italics added). These statements are correct in spirit but slightly confusing. The total payment for both claimants may exceed the estate. We suggest that the second sentence in the quote should be changed to: "If each claimant is awarded an amount such that the total payment exceeds the estate under bankruptcy; the excess is deducted in equal shares from the claimants' *de facto* debt, $\min(E, d_i)$" (underlined words edited into the original Footnote).

	$\tilde{d_1}$	$\tilde{d_2}$	$\tilde{d_1}+\tilde{d_2}-E$	$s_1^L = \tilde{d_1}-(\tilde{d_1}+\tilde{d_2}-E)/2$	$s_2^L = \tilde{d_2}-(\tilde{d_1}+\tilde{d_2}-E)/2$
$E \le d_1 \le d_2$	E	E	E	$E/2$	$E/2$
$d_1 \le E \le d_2$	d_1	E	d_1	$d_1/2$	$E-d_1/2$
$d_1 \le d_2 \le E$	d_1	d_2	d_1+d_2-E	$(E+d_1-d_2)/2$	$(E+d_2-d_1)/2$

Note: $\tilde{d_1} = \min\{d_1, E\}$; $\tilde{d_2} = \min\{d_2, E\}$.

Table 3. 2-Creditor Bankruptcy Allocation (s_1^L, s_2^L) under Equal Sharing of De Facto Total Loss (Alternative definition)

Comparing the corresponding entries in the last two columns of Tables 2[29] and 3 confirms that for all bankrupt estates E with individual claims d_1 and d_2, we have:

$$s_i^G = \tilde{c_i} + (E-\tilde{c_1}-\tilde{c_2})/2 = s_i^L = \tilde{d_i} - (\tilde{d_1}+\tilde{d_2}-E)/2.$$

Thus, the proposed requirement of equal sharing of *de facto* total loss and the Aumann and Maschler (1985) requirement of equal sharing of *de facto* contested claim lead to the same allocations assigned to the two creditors. While this equivalence was alluded to previously and may not be surprising, we will continue to analyze these approaches through a different angle to make clear that the two alternatives are but different views of the same coin, one inspecting it from the top and the other from the bottom. Before that, note that taking the creditors' claims as fixed, each creditor's share is an increasing (but not strictly increasing) function of E. When the available funds of the bankrupt estate increase, each creditor should expect to recover more, and certainly not less, of her money. Also, it is intuitive and embedded in the solutions that a creditor with a larger claim should receive no less a repayment than a creditor with a smaller claim.

Besides the identical bankruptcy allocations resulting from the alternative requirements, two important relations embedded in Tables 2 and 3 can be confirmed. The first and more important relation is that the *de facto* concession to claimant i (from j) and the *de facto* claim for creditor j are E-complements. To observe that, concentrate on a specific range of the estate E (say the second row). Note the sum of the variable in column 1 of one Table and the variable in column 2 of the other Table always equals E. More specifically, $\tilde{c_1}+\tilde{d_2}=E$ and $\tilde{c_2}+\tilde{d_1}=E$; we shall refer to these as the E-complementarity conditions. While these relations may be slightly harder to detect across the two Tables, they are almost definitional. Consider first the case in which the claim of creditor j is less than the estate value ($d_j<E$). The *de facto* concession to i from j is then positive and $\tilde{c_i}=\max\{E-d_j,0\}=E-d_j$, while the *de facto* claim of j is the nominal claim, $\tilde{d_j}=\min\{d_j, E\}=d_j$. This renders $\tilde{c_i}+\tilde{d_j}=E$. In the case in which the claim of creditor j is greater than the estate value ($d_j>E$), the *de facto* concession to i from j is zero, $\tilde{c_i}=\max\{E-d_j,0\}=0$, and the *de facto* claim of j is reduced to the value of the estate, $\tilde{d_j}=\min\{d_j, E\}=E$. Again, these values confirm the complementary relation of $\tilde{c_i}$ and $\tilde{d_j}$ within the range of E: $\tilde{c_1}=E-\tilde{d_2}$ and $\tilde{c_2}$

[29] It is easy to confirm that the shares s_1^G, s_2^G in Table 2 match the description in Aumann and Maschler (1985, p.198).

$=E–d_1^{\sim}$. With *de facto* refinements, the concession due one creditor is the residual of the estate over the other's claim. Roughly, the complementary relation is much like j having a piece of E-pie. If j is not very hungry and the pie is too big, she offers what she cannot eat to i. If j is very hungry and the E-pie is too small to satisfy her appetite, she can only consume the whole pie herself and offers nothing to i (but she cannot unilaterally force i to give her some of i's pie).

Another important relation easily observed from Tables 2 and 3 is that the total gains beyond the *de facto* concessions from the bankrupt estate, $E–c_1^{\sim}–c_2^{\sim}$, and the total *de facto* loss due to bankruptcy, $d_1^{\sim}+d_2^{\sim}–E$, are always equal. That is, the recoverable gains to be equally shared by creditors, as specified in Aumann and Maschler (1985), can be thought of as an irretrievable loss to be borne equally by both. The alternative interpretations can be deduced easily from the pair of E-complementarity relations: summing $c_1^{\sim}+d_2^{\sim}=E$ and $c_2^{\sim}+d_1^{\sim}=E$ gives $c_1^{\sim}+c_2^{\sim}+d_1^{\sim}+d_2^{\sim}=2E$; rearranging this equation gives $E–c_1^{\sim}–c_2^{\sim}=d_1^{\sim}+d_2^{\sim}–E$. It is notable that the recoverable *de facto* gain and its alternative interpretation, the irretrievable *de facto* loss, are always positive. (They take the value of either the estate or the smaller claim if the bankrupt estate is less than at least one creditor's claim, and they equal $d_1+d_2–E$ when the estate is greater than both claims, in which case it is positive because of bankruptcy.)

It is noteworthy that the E-complementarity conditions imply that the total *de facto* loss $d_1^{\sim}+d_2^{\sim}–E$ can be interpreted as $d_1^{\sim}–c_1^{\sim}$ or $d_2^{\sim}–c_2^{\sim}$. These two results illustrate something that is intuitive: the maximum potential (*de facto*) loss for each creditor is the same if the creditor has to bear the burden alone. It is given by the (*de facto*) total loss facing both creditors together, $d_1^{\sim}+d_2^{\sim}–E$. Since $E–c_1^{\sim}–c_2^{\sim}=d_1^{\sim}+d_2^{\sim}–E$ is always positive under bankruptcy, we now have the following:

$$E–c_1^{\sim}–c_2^{\sim} = d_1^{\sim}+d_2^{\sim}–E = d_1^{\sim}–c_1^{\sim} = d_2^{\sim}–c_2^{\sim} > 0.$$

These relations indicate that the distance from $c_i^{\sim}$ to $d_i^{\sim}$ is $E–c_1^{\sim}–c_2^{\sim}$, and this distance can also be written as $d_1^{\sim}+d_2^{\sim}–E$. The Aumann and Maschler (1985) definition identifies the share assigned to creditor i as adding half the distance between $c_i^{\sim}$ and $d_i^{\sim}$ to the lower bound $c_i^{\sim}$; our new definition presents the share to creditor i as subtracting half the distance between $c_i^{\sim}$ and $d_i^{\sim}$ from the upper bound $d_i^{\sim}$. Naturally, the two definitions are equivalent since the mid-point between $c_i^{\sim}$ and $d_i^{\sim}$ is identified in both alternatives. Thus, when viewed appropriately, each creditor may consider that she is awarded half the potential recovery or she is made to suffer half the total loss from bankruptcy; the amounts she receives are the same:

$$s_1–c_1^{\sim} = d_1^{\sim}–s_1 \text{ and } s_2–c_2^{\sim} = d_2^{\sim}–s_2.$$

As they share an equal amount of gain in the Aumann and Maschler approach, and they suffer an equal amount of loss in the alternative approach, we end up with the following:

$$s_1–c_1^{\sim} = d_1^{\sim}–s_1 = d_2^{\sim}–s_2 = s_2–c_2^{\sim}.$$

In other words, no matter which creditor we look at, and whether we view the partial repayment from the bankrupt estate as an individual *de facto* net gain or *de facto* net loss, under the CC allocation principle they are all the same!

4. Gains and Losses Under the CCC bankruptcy allocations

That individual *de facto* net gain and individual *de facto* net loss are equal for a creditor and across creditors, whatever the size of the estate as long as there is bankruptcy, is an important distinguishing feature for CCC allocations and a fact not well-observed in the literature. Instead, the literature centers its attention on the related and perhaps more intuitive concepts of recovery and forfeiture. It stresses the sizes of nominal gains s_i (the actual repayment without any qualification) and of nominal losses $d_i - s_i$ (the actual difference between the creditor's claim and the repayment) across creditors.[30] Table 4 below tabulates the relative magnitudes of these nominal concepts, where creditor 1's nominal claim is assumed to be less than creditor 2's nominal claim; it also presents the value of the *de facto* net gain $s_i - c_i^{\sim}$, which is identical across individual creditors and equals their *de facto* net loss $d_i^{\sim} - s_i$ in the last column.

	s_1	s_2	$d_1 - s_1$	$d_2 - s_2$	$s_1 \leftrightarrow s_2$	$d_1 - s_1 \leftrightarrow d_2 - s_2$	$s_i - c_i^{\sim} = d_i^{\sim} - s_i$
$E \le d_1 \le d_2$	$E/2$	$E/2$	$d_1 - E/2$	$d_2 - E/2$	$s_1 = s_2$	$d_1 - s_1 \le d_2 - s_2$	$E/2$
$d_1 \le E \le d_2$	$d_1/2$	$E - d_1/2$	$d_1/2$	$d_2 - (E - d_1/2)$	$s_1 \le s_2$	$d_1 - s_1 \le d_2 - s_2$	$d_1/2$
$d_1 \le d_2 \le E$	$(E+d_1-d_2)/2$	$(E+d_2-d_1)/2$	$(d_1+d_2-E)/2$	$(d_1+d_2-E)/2$	$s_1 \le s_2$	$d_1 - s_1 = d_2 - s_2$	$(d_1+d_2-E)/2$

Table 4. Comparison of nominal gains, nominal losses, and de facto gains (same as de facto losses) under CC allocations

The first four columns in Table 4 present the nominal gains and the nominal losses for different ranges of values of the estate E. The next two comparison columns, along with the assumption that d_1 is less than d_2, indicate that nominal gains and nominal losses are order preserving. That is, under the CC bankruptcy division, the nominal gain for a creditor with a lesser claim is less than (or equal to) the nominal gain for a creditor with a greater claim: $d_1 \le d_2 \Rightarrow s_1 \le s_2$. Likewise, a lesser creditor loses less than (or equal to) what a greater creditor would lose under the CC bankruptcy division: $d_1 \le d_2 \Rightarrow d_1 - s_1 \le d_2 - s_2$.[31] Note that these nominal gains and nominal losses are order-preserving but not strictly order-preserving. In some regions, a creditor with a smaller claim receives the same nominal gain as a creditor with a larger claim.[32] In other

[30] Aumann and Maschler (1985, Footnote 18) states, "Specifically, whether we think of the outcome to Creditor *i* as an award of s_i or a loss of $d_i - s_i$." (The mathematical expressions have been adjusted to the notation used in this article.) This starts the trend in the literature to focus on nominal gain and the related nominal loss. Although it is correct in spirit, our analysis shows that it is more systematic to consider the outcome to creditor *i* as a *de facto* gain $s_i - c_i^{\sim}$ or, equivalently, as a *de facto* loss $d_i^{\sim} - s_i$.

[31] Although we only illustrate results in the case of two creditors, applying the logic to any pair of creditors in the general case of *n* creditors supports our conclusion. While they did not expand on the point as we do here, this order-preserving property was pointed out in Aumann and Maschler (1985, p.205).

[32] For example, when $E < d_1 < d_2$, $s_1 = s_2 = E/2$, creditor 1 with a smaller claim receives the same nominal award as creditor 2 with a larger claim.

regions, a creditor with a smaller claim incurs the same nominal loss as a creditor with a larger claim.[33] The last column in Table 4 reiterates our findings in the previous section that all individual *de facto* net gains and *de facto* net losses are equal across all individuals. The exact amounts can be readily computed from Table 2.

It is not clear whether the equal nominal gain feature for small estates and the equal nominal loss feature for large estates inspired Aumann and Maschler. In proving that a unique CCC solution exists for every bankruptcy problem, Aumann and Maschler (1985, p.200) describe how the allocation evolves when E changes in two separate regions. When E is less than half the total claim, attention is centered on the equality of the creditor shares with small E and how an additional dollar gain is shared by the creditors as E increases. When E exceeds half the total claim, a mirror image of the previous process focuses on equality of individual loss when the estate is very large and how an additional dollar of total loss is borne by the creditors as E decreases. In other words, individual gains s_i are considered when E is below half the total claim while individual losses $d_i - s_i$ are contemplated when E exceeds half the total claim.[34]

The order preserving feature of both nominal gains and nominal net losses is in stark contrast with what is uncovered in the last section. Behind the CCC principle, with appropriate and intuitive restrictions on the basic variables, namely that a (*de facto*) concession $c_i^{\sim}$ cannot be negative and a (*de facto*) claim $d_i^{\sim}$ cannot exceed the estate value, the *de facto* net gains awarded to the lesser creditor and to the greater creditor are equal: $s_1 – c_1^{\sim} = s_2 – c_2^{\sim}$. Likewise, the *de facto* net losses suffered by the lesser and the greater creditors are also identical: $d_1^{\sim} – s_1 = d_2^{\sim} – s_2$. Further, the *de facto* net gain for any creditor i equals the *de facto* net loss for any creditor j: $s_i – c_i^{\sim} = d_j^{\sim} – s_j$. These results come from the fact that under CCC division, a pair of creditors share total *de facto* gains equally and also suffer the total *de facto* loss equally. Thus, fundamentally, equal sharing is the name of the game for CCC allocations, whether we think in terms of gains or loss in the restricted *de facto* fashion. Looking a bit deeper, this means that one creditor counts just as much as another creditor, while attention is paid to the fact that an outstanding claim exceeding the estate has no role in the repayment assignment.

In describing the CCC allocation explicitly, Aumann (2002) separates it into two parts. In the first part, "When the estate does not exceed half the sum of the claims, each woman gets the same amount, so long as this does not exceed half her claim." This part excludes scenarios in which $d_1 \leq d_2 \leq E$ hold, since E would exceed half of the total claim otherwise. For the remaining two scenarios, if $E \leq d_1 \leq d_2$ (the first row in Table 4) both creditors always receive the same amount (E/2), in which case their award does not exceed half their individual claim. When

[33] For example, when $d_1 < d_2 = E$, $s_1 = d_1/2$, $s_2 = d_2 – d_1/2$, $d_1 – s_1 = d_1/2$, and $d_2 – s_2 = d_1/2$. Creditor 1 with a smaller claim suffers the same nominal loss as creditor 2 with a larger claim.

[34] After the full description of the CCC allocations, Aumann and Maschler (1985) also provides an alternative description in terms of increasing shares from additional E for the case in which E exceeds half the total claim.

$d_1 \le E \le d_2$ (the second row in Table 4), each creditor receives the same amount ($d_1/2$) only if $d_1 = E$. Combining these observations, each woman receives the same amount as long as $E \le d_1$, or, as long as the estate is less than both claims. Thus, creditors receive the same award only when $E \le d_1$ but they receive identical *de facto* gains all the time. This is because when $E \le d_1$ and thus $E \le d_2$ as well, there are no concessions to speak of, and individual awards s_i are equal as each award represents half of the total gain, which is the individual *de facto* net gain $s_i - c_i^{\sim}$.[35] When E exceeds d_1, positive but different concessions start kicking in and are added to half of the total gain, destroying equality of the awards if the creditor claims are unequal. As it happens, the award to the greater d_2-creditor is larger because the amount conceded to her ($E–d_1$) is larger than the amount conceded to the lesser d_1-creditor ($E–d_2$), while the *de facto* net gains remain equal as they represent the equal sharing part of total gains.

To complete describing the remaining part of the CCC allocation, Aumann (2002) states, "When the estate does exceed half the sum of the claims, the calculation is made in accordance with each woman's loss: the difference between her claim and what is actually paid out to her. The rule is that all the creditors lose the same amount, so long as none of them loses more than half her claim." Here, the requirement that the estate exceeds half the sum of the claims is incompatible with $E \le d_1 \le d_2$. In the other cases where $d_1 \le d_2 \le E$ (the third row in Table 4), both creditors lose an equal amount $(d_1+d_2-E)/2$, and it is easy to confirm that each creditor loses less than her claim. In cases where $d_1 \le E \le d_2$ (the second row in Table 4), each creditor loses the same amount ($d_1/2$) as long as $E = d_2$. Thus, each creditor incurs the same loss only when $E \ge d_2$; but they bear identical *de facto* losses all the time. This is because when $E \ge d_2$ and thus $E \ge d_1$ as well, each creditor's nominal claim d_i is her *de facto* claim $d_i^{\sim}$, making her nominal loss $d_i - s_i$ and her *de facto* loss $d_i^{\sim} - s_i$ one and the same;[36] each represents half of the *de facto* total loss $d_1^{\sim}+d_2^{\sim}-E$. When E falls below d_2, the difference between the nominal claim and the *de facto* claim for the greater creditor is larger than the difference for the lesser creditor ($d_2-d_2^{\sim} > d_1-d_1^{\sim}$), destabilizing equality between the nominal losses if creditor claims are different. As E is below d_2, $d_2-d_2^{\sim} > d_1-d_1^{\sim}$ is equivalent to $(d_2-s_2)-(d_2^{\sim}-s_2) > (d_1-s_1)-(d_1^{\sim}-s_1)$. As $d_2^{\sim}-s_2$ and $d_1^{\sim}-s_1$ both equal half of the *de facto* total loss, this leads to $d_2-s_2 > d_1-s_1$, a larger loss for the greater d_2-creditor. Thus, the larger loss of the greater creditor comes from her relatively larger gap between money d_2 that is not all there and the value E of the estate. ($d_2-E>d_1-E$ when $E<d_i$ implies that $d_2-d_2^{\sim} > d_1-d_1^{\sim}$). But *de facto* net losses remain identical since they represent the equal sharing part of the total loss.

To recap, while the literature stresses that CCC bankruptcy division requires equal sharing of total gains between any two creditors, it pays considerable attention to the relative gains s_i and the relative losses $d_i - s_i$ across creditors. In particular, while the gains for the two creditors are

[35] This can be confirmed by comparing the first, second, and last cells on the first row in Table 4.

[36] This can be confirmed by comparing the third, fourth, and last cells on the last row in Table 4.

equal at times and their losses are equal at other times, equalities only hold *some of the time*. Further, the gain for a creditor with a lesser claim is no greater than the gain for a creditor with a greater claim, and the loss for this creditor with a lesser claim is also no greater than the loss for a greater creditor. The fact that the gains and losses are order-preserving with respect to the sizes of creditors' claims is comforting; after all, those with a greater claim ought to be repaid no less than those with a lesser claim. But this study points out that other important equalities do hold *all the time*. Namely, the equalities of *de facto* net gains $s_i - c_i^{\sim}$ and of *de facto* net losses $d_i^{\sim} - s_i$ should be brought to bear instead of the gains and losses in general. The fact that equal-sharing of the total gains and equal-sharing of the total loss are equivalent in the refined *de facto* sense is rather surprising. Importantly, it provides us with a deeper understanding of the fundamental requirement behind the CCC principle.

5. The CCC Allocations for the Building-Block Case of Two Creditors

While the equal sharing principle of contested claim CC is straightforward, to implement and find the CCC division may not be straightforward, especially when it involves more than two creditors.[37] As noted earlier in the proof for the unique existence of a CCC division, Aumann and Maschler (1985) characterize the bankruptcy division, but they do not provide a detailed discussion of how this characterization comes about. When there are multiple creditors, any subset of creditors, including any two creditors, must follow the equal sharing principle of contested claim to split their award sum. This consistency requirement suggests some recursive steps are involved, and ultimately the bankruptcy division of two creditors underlies the allotments for any number of creditors. Tables of CCC divisions have not been readily available even for a small number of creditors, but would further our understanding.[38] To that end, this section characterizes the divisions for any two creditors, the building blocks for all to come. The next section pursues the allotments for three creditors; the investigation highlights the recursive structure of the CCC divisions in general.

The CCC divisions for two creditors were presented earlier in Tables 2 and 3, with the goal of confirming that two alternative ways of defining CCC divisions are equivalent. To identify the

[37] Aumann (2002, p.6) notes, "Indeed, we are not dealing here with a method but, rather, with a condition. Given a certain division, one may check whether or not that division is CG-consistent. However, it may not be clear, at the outset, how one arrives at a CG division." Note that "CG," Contested Garment in Aumann, has been renamed "CC," Contested Claim, in this article.

[38] Kaminski (2000) provides an ingenious way to visualize how an increase in the amount of a bankrupt estate should be divided among all creditors. But the approach suffers from the lack of explicit connection between the mathematical solution and the visual solution. For example, with a visual solution for two creditors, one needs to compute backward how E should be related to d_1 and d_2. Going the other way, even in the case of three creditors, one may have to test a few visual solutions in order to find the correct one that leads to the mathematical form of the CCC division. This section organizes and lists all the CCC divisions as functions of the estate with fixed claims, but we did not enlist the help of visual solutions.

precise CCC allocation, given the claims of the creditors and the estate, Table 5 provides a useful alternative. It pays more attention to the boundaries where different types of divisions are called for across boundaries, and the boundary columns are highlighted. Strictly speaking, an estate value half the sum of the claims is not a boundary case. But since the literature stresses the difference in the estate allocation when the estate is above or below half the total claim, it is treated as a boundary case as well. To separate the types of divisions from one scenario to the next, it is convenient to redefine the estate by using the claims and additional parameters. The calculation of the shares assigned to the two creditors is straightforward. Appendix I presents the shares in a natural way, depending on the claims and the value of the estate E; they are then rewritten to depend on the claims and other parameters. This second presentation in Table 5 makes it easier to find the precise CCC allocation given fixed values for the claims and the estate; all parameters α, β, β', and α' are positive, with upper bounds clearly specified.

	1	2	3	4	5	6	7	8
E / Claim	2α ($\alpha \le d_1/2$)	d_1	$d_1+\beta$ ($\beta \le (d_2-d_1)/2$)	$(d_1+d_2)/2$	$(d_1+d_2)/2+\beta'$ ($\beta' \le (d_2-d_1)/2$)	d_2	$d_2+2\alpha'$ ($\alpha' \le d_1/2$)	d_1+d_2
d_1	α	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2+\alpha'$	d_1
d_2	α	$d_1/2$	$d_1/2+\beta$	$d_2/2$	$d_2/2+\beta'$	$d_2-d_1/2$	$d_2-d_1/2+\alpha'$	d_2

Table 5. 2-Creditor Bankruptcy Shares Under Principle of Equal Division of Contested Claim ($d_1<d_2$)

Table 5 confirms equal sharing of E between the two creditors as long as the estate is less than or equal to the smaller claim d_1 (columns 1 and 2, before crossing the first boundary). As the estate increases beyond d_1 but remains less than d_2, the share assigned to the lesser creditor remains at $d_1/2$, and any extra increase in the value of the estate goes to repay the greater creditor. In this region where $d_1 < E < d_2$, the *de facto* concession offered to creditor 1 is zero, as the estate has not caught up with d_2 yet and no concession is offered by creditor 2, but the *de facto* concession given to creditor 2, $E-d_1$, increases as the estate E increases (β or β' increases in Table 5). Meanwhile, the *de facto* total gain to be equally shared by both creditors holds still at $E-(E-d_1) = d_1$. This is why the share given to creditor 1 stays at $d_1/2$, while the share given to creditor 2 keeps on increasing; the increase for creditor 2 is purely due to the higher conceded amount offered to him. Finally, note that Table 5 shows that when the bankrupt estate value increases beyond d_2 (column 7, beyond the second boundary), both creditors share the *extra* amount of the estate beyond d_2 equally. In this region where $d_1 \le d_2 \le E$, the amount to be shared by the two creditors is d_1+d_2-E,[39] which takes the form of $d_1+d_2-(d_2+2\alpha')=d_1-2\alpha'$. Adding half of this amount to the concessions due to creditors 1 and 2, $2\alpha'$ and $d_2-d_1+2\alpha'$, respectively, results in

[39] This was shown in the last row in Table 2.

the shares shown in column 7. This column also implicitly confirms that both creditors' losses are equal. When $d_1 < d_2 < E$, the *de facto* total loss and the nominal total loss are the same and take the value of $d_1+d_2-E = d_1+d_2-(d_2+2\alpha') = d_1-2\alpha'$.[40] This total loss, which of course equals the total gain shown a little earlier, is equally shared by the two creditors, leaving each creditor with a loss of $d_1/2-\alpha'$.

One final note not observed before is that when the value of the estate is half the total claim, each creditor receives half her claim; it turns out that this exceptional case is the only case where CCC repayments to the creditors also satisfy the proportionality principle.[41] The presentation of shares in Table 5 is limited to two creditors and the bankruptcy allocation mechanism should be extended to more creditors. To this we now turn in the next section.

6. The CCC Allocations for Three Creditors and the Nathan Examples

Following the full description of how the bankrupt estate is divided between two creditors under equal sharing of contested claim, we now concentrate on the division patterns for the case of three creditors for any value of estate E with fixed claims d_1, d_2, and d_3. The proof in Appendix II shows explicitly how the CCC allocation for three creditors is found by consistently applying the equal division of contested claim to any two creditors, making use of results found in the 2-creditor case. This proof clearly illustrates the recursive structure behind the division patterns for any number of creditors.[42] Table 6 below duplicates the results in Table A2 of Appendix II, with highlights added to boundary cases in columns 2, 4, 8, and 10.

[40] Alternatively, $s_2-s_1 = d_2\tilde{}-d_1\tilde{} = d_2-d_1$. The first equality comes from definitional manipulation, and the second equality holds because $d_1 \leq d_2 \leq E$; both properties can be observed in Table 3. Rewriting, we have $s_2-d_2 = s_1-d_1$.

[41] For the proof see Fon (2016*a*).

[42] As alluded to earlier, in their proof of the existence of a consistent solution to a bankruptcy problem, Aumann and Maschler characterize the CCC solution explicitly, but the characterization does not clearly illustrate the recursive nature and the consistency requirement involved.

	1	2	3	4	5	6	7	8	9	10	11	12
E \ Claim	3α, ($\alpha \le d_1/2$)	$3d_1/2$	$3d_1/2+2\beta$, ($\beta \le (d_2-d_1)/2$)	$d_1/2+d_2$	$d_1/2+d_2+\gamma$, ($\gamma \le (d_3-d_2)/2$)	$(d_1+d_2+d_3)/2$	$(d_1+d_2+d_3)/2+\gamma'$, ($\gamma' \le (d_3-d_2)/2$)	$d_3+d_1/2$	$d_3+d_1/2+2\beta'$, ($\beta' \le (d_2-d_1)/2$)	$d_2+d_3-d_1/2$	$d_2+d_3-d_1/2+3\alpha'$, ($\alpha' \le d_1/2$)	$d_1+d_2+d_3$
d_1	α	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2+\alpha'$	d_1
d_2	α	$d_1/2$	$d_1/2+\beta$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2/2+\beta'$	$d_2-d_1/2$	$d_2-d_1/2+\alpha'$	d_2
d_3	α	$d_1/2$	$d_1/2+\beta$	$d_2/2$	$d_2/2+\gamma$	$d_3/2$	$d_3/2+\gamma'$	$d_3-d_2/2$	$d_3-d_2/2+\beta'$	$d_3-d_1/2$	$d_3-d_1/2+\alpha'$	d_3

Table 6. 3-Creditor Shares Under CCC Principle **($d_1<d_2<d_3$)**

With three creditors, the number of boundary cases expands to four from two in the case of two creditors. Observe that whenever the estate is no greater than $3d_1/2$ (columns 1 and 2), all creditors share the estate equally. Between the first and second boundaries in which $3d_1/2 < E \le d_1/2+d_2$ (columns 3 and 4), the share of creditor 1 stays constant at $d_1/2$, while creditors 2 and 3 split the amount of E beyond $3d_1/2$. The higher shares for creditors 2 and 3 come from the amounts conceded by creditor 1 to each of them (under a two person setting). Likewise, between the second and third boundaries, whenever a creditor's share is higher than another creditor, the difference comes from the concession offered to the former by the latter (columns 5 to 8). Between the third and fourth boundaries, the sharing characteristics between creditors 1 and 2 and between 1 and 3 are the familiar one in the last scenario, where the joint sum lies between the pair of creditor claims (columns 9 and 10). However, the combined awards for creditors 2 and 3, $d_3+2\beta'$ and $d_3+d_2-d_1$, exceed both claims d_3 and d_2, resembling the relations between the estate $d_2+2\alpha'$ and claims d_2 and d_1 in column 7 of Table 5. There we found that creditors 1 and 2 equally share any extra amount beyond d_2, indicating that here creditors 2 and 3 share any extra amount beyond d_3 ($2\beta'$ and d_2-d_1). Lastly, beyond the fourth boundary (column 11), all three creditors incur the same loss of $d_1/2-\alpha'$.[43]

[43] We should note that results from Table 6 are consistent with the presentation of the CCC allocation in Aumann and Maschler (1985, p.200) where they present but do not explain the CCC allocations. "To show that there is at least one consistent solution, we exhibit it as a function of the estate E (for fixed debts $d_1, ..., d_n$). Let us think of the estate as gradually growing. When it is small, all n claimants divide it equally. This continues until 1 has received $d_1/2$; for the time being she then stops receiving payments, and each additional dollar is divided equally between the remaining n–1 claimants. This, in turn, continues until 2 has received $d_2/2$, at which point she stops receiving payments for the time being, and each additional dollar is divided equally between the remaining n–2 claimants. The process continues until each claimant has received half her claim. This happens when $E = D/2$, where $D = d_1 + ... + d_n$ = the total debt." The description continues after one paragraph but we leave it to readers to confirm our results in the second half of the table. Thus, the pattern explained in Aumann and Maschler, when applied to the case of three creditors, matches exactly what is proved in the appendix of this paper.

The CCC divisions in Table 6 are given in terms of gains from threshold values.[44] In fact, one can imagine that individual allotments are computed by the equal sharing of the contested claims approach. Following Aumann and Maschler's description of the CCC allocations, the literature tends to use alternative approaches to characterize the CCC divisions when the estate is less than half the total claim versus when the estate is greater than half the total claim. When the estate is small, attention is centered on nominal gain sharing allotted to different creditors. But when the estate is large, attention is paid in terms of nominal loss sharing.[45] To help clarify the aspect of loss sharing when the estate is large, at the end of Appendix II, Table A3 reformats the second right half of Table 6 with different threshold values so that the shares given to different creditors are given in terms of losses.[46] In sum, Table A3 gives an alternative presentation to Table 6 which makes very clear the symmetry of the CCC divisions with respect to half of the total debt as estate value varies. This Table illustrates the self-dual property of the CCC allocation rule where a self-dual rule treats losses and awards in the same way.

Comparing Tables 5 and 6, a few interesting properties emerge. First, the first two rows in Table 5 are incorporated in Table 6, where the divisions for creditors 1 and 2 remain the same. This is to be expected, since consistency in CCC allocations means that taking creditors 1 and 2 as a subgroup, their allocation has to follow the basic principle of equal division of contested claim incorporated in Table 5. This is the result of the recursive nature of the problem mentioned earlier. Second, in terms of the smallest claim d_1, the length of the interval for equal gains increases as the number of creditors against the estate increases. In particular, the length of the equal-gain interval for two creditors is d_1, the length for three creditors is $3d_1/2$, and the length for n creditors is $nd_1/2$.[47] On the other hand, as a ratio of the total claim against the estate, the interval of *unequal* gains grows as the number of creditors grows. This is to be expected, since the requirement of equal shares for any two creditors, which implies equal shares for all creditors, becomes harder to fulfill as the number of creditors grow. Likewise, the behavior for equal losses follows a similar pattern. The length of the equal-loss interval for two creditors is

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[44] In Table 6, parameters α, β, γ, γ', β', and α' are used to bring out the symmetries in the allocation pattern before and after half of the total debt.

[45] As mentioned, *supra* note 43, when presenting the CCC allocation, Aumann and Maschler (1985, p.200) first describe how an additional amount of E is divided among claimants when $E \leq D/2$. Before they continue the discussion of how an additional amount of E is divided when E increases further, Aumann and Maschler follow immediately with: "When $E \geq D/2$, the process is the mirror image of the above. Instead of thinking in terms of i's award s_i, one thinks in terms of her loss $d_i - s_i$, the amount by which her award falls short of her claim. When the total loss $D–E$ is small, it is shared equally between all creditors, so that creditor i receives her claim d_i less $(D–E)/n$." (The award variable has been adjusted to the s_i notation used in this article.) They continue to describe how additional losses should be shared among creditors.

[46] In Table A3, parameters α, β, γ are enlisted twice to illustrate that an individual gain in a region before half of the total debt matches exactly the loss in a corresponding region after half of the total debt.

[47] In general, let $d_1 \leq d_2 \leq \ldots \leq d_n$. Table 5 shows that equal shares between creditor 1 and any other creditor i ($s_1 = s_i \equiv s$) implies that $s \leq d_1/2$. The total amount satisfying equal shares for n creditors is then ns. Since $ns \leq n{\cdot}d_1/2$, the interval of equal gains for all creditors is from 0 to $n{\cdot}d_1/2$.

$(d_1+d_2)–d_2=d_1$, for three creditors is $(d_1+d_2+d_3)–(d_2+d_3–d_1/2)=3d_1/2$, and for n creditors is $n\cdot d_1/2$. Thus, in terms of the smallest claim d_1, the length of the interval for equal losses increases as the number of creditors against the estate increases. But in terms of the total claim against the estate, the interval of unequal losses grows when there are more creditors. Here, readers are reminded that behind the CCC allocations, the *de facto* net gain and the *de facto* net loss are always equal across all creditors.

As the literature pays much attention to the equal shares and equal losses regions, we conclude that in general, in the case of n creditors where the first creditor's claim d_1 is the smallest, whenever $E \le n\cdot d_1/2$, all creditors receive the same share. At the other end of the spectrum, whenever $E \ge \sum d_i – n\cdot d_1/2$, all creditors share the deficit equally.[48]

Next, we turn to the historically important Talmudic examples, where the claims of the creditors are fixed, but the estate value changes from case to case. Taking the general claims in Table 6 to be the specific claim values used in the Nathan examples, Table 7 incorporates the numerical results provided by Nathan, where E equals 100, 200, and 300. In particular, when the estate is worth 100 (column 1), equal sharing for all is called for. When the value of the estate is 200 (β = 25 in column 3), the least creditor receives 50, and each of the other two creditors receives an equal share of 75. When the estate is worth 300 (column 6), proportional division results.

	1	2	3	4	5	6	7	8	9	10	11	12
E / d_i	3α ($\alpha\le 50$)	150	$150+2\beta$ ($\beta\le 50$)	250	$250+\gamma$ ($\gamma\le 50$)	300	$300+\gamma'$ ($\gamma'\le 50$)	350	$350+2\beta'$ ($\beta'\le 50$)	450	$450+3\alpha'$ ($\alpha'\le 50$)	600
100	α	50	50	50	50	50	50	50	50	50	$50+\alpha'$	100
200	α	50	$50+\beta$	100	100	100	100	100	$100+\beta'$	150	$150+\alpha'$	200
300	α	50	$50+\beta$	100	$100+\gamma$	150	$150+\gamma'$	200	$200+\beta'$	250	$250+\alpha'$	300

Table 7. CCC Shares for Nathan's Example in the Talmud

Table 7 extends different numerical tables given in the literature.[49] With these tabulations, we can calculate the division for any value of the estate with fixed claims 100, 200, and 300, thereby providing a complete and explicit answer to the two-thousand year old Talmudic question. For

[48] Coincidentally, in the third justification presented in Aumann and Maschler (1985, p.206), the coalitional procedure requires three branches to divide the amount among creditors. In particular, when $E \le n\cdot d_1/2$, equal awards are assigned to all creditors; when $E \ge \sum d_i – n\cdot d_1/2$, all creditors are assigned equal losses. Our discussion in this section provides intuition why the two branches should be divided as described, while Aumann and Maschler only prescribe how a CCC division could be found without giving any explanation.

[49] Table 7 incorporates Table 6 in Aumann (2002), which provides the shares assigned to the three creditors as a function of the entire estate, by increments of 5. This table also generalizes table II, Appendix A, in Elishakoff and Begin-Drolet (2009), which provides each assigned share to a creditor as a function of the estate value, by increments of 10.

example, when the estate value is 400, column 9 (with $\beta'=25$) indicates that the 100-, 200-, and 300-claim creditors receive 50, 125, and 225 respectively.[50]

Lastly, it should be mentioned that the CCC allocations found so far clearly should be extended to any number of creditors. But doing so would be very tedious, conceptually not helpful, and beyond the scope of this article.

7. Conclusion

This paper takes Aumann and Maschler's brilliant discovery of the game theoretic connection with the 2 millennia old Talmudic puzzle as a starting point and provides a fuller picture on what is behind the bankruptcy solutions. It expands on the fact that, with appropriate bounds, the underlying divisional CCC principle can be thought of as equal sharing of losses as well as gains, whatever the value of the bankrupt estate. It also tabulates and proves bankruptcy allocations for the basic cases of 2 and 3 creditors, information not readily available in the existing literature.

Specifically, the CCC principle implicitly incorporated in the Talmudic numerical examples is built on the basic case of 2 creditors. Aumann and Maschler defined it as the principle of equal sharing of contested claim. Each creditor is first awarded the amount conceded by the other creditor, the *de facto* concession; it is positive if the other creditor's claim is less than the estate's value and is zero otherwise. Each creditor then gets an additional amount which is half of the residual of the *de facto* concessions from the estate. Much like *de facto* concession cannot be negative, *de facto* claim cannot exceed the estate value. This article studies the alternative view of the equal sharing of total loss, where total loss is the difference between *de facto* claims of both creditors and the value of the estate. It is shown that the two approaches are equivalent. This means that the CCC principle splits everything down the middle, whether creditors view the estate as a gain or recognize the deficit of the estate from total claims as a loss.

In the early days of the development of behavior economics, Kahneman, Knetsch, and Thaler (1986) illustrated that judgments of fairness are often influenced by framing and reference points, which by now is part of the conventional wisdom in economics. Reference points help define perceptions of gains and losses by individuals; different approaches to framing thereby might affect peoples' views of fairness. This article points out that the CCC allocations are not subject to framing manipulation as long as creditors recognize that money absent from the estate should not be counted as part of their claims because there is nothing to share. Reference points and framing do not matter. In fact, when a creditor compares her share with that of another creditor, and if she thinks of the repayment as a gain, she shares the contested claim equally with

[50] Aumann (2002, p.9) used this specific value for the estate to describe the same CCC division as here.

the other creditor.[51] If she thinks of the deficient estate as a loss, she shares it equally with her counterpart too.[52] Further, if she compares her individual gain and individual loss, they are also the same.[53]

The CCC principle has both ancient backing from the Talmud and modern support from the cooperative game solution of nucleolus. The brilliance of the resulting CCC division actually lies heavily on the consistency requirement that for any pair (and more generally, any group) of creditors, their shares must satisfy the CC principle of equal sharing of contested claim. To extend the CC principle consistently to any number of creditors is very sensible. In the game theoretical setting to find nucleolus, a coalitional game corresponding to the bankruptcy problem with a set of creditors is formulated. The goal is to find an allocation that minimizes the worst inequality. Or, for each coalition of creditors, the dissatisfaction with the proposed allocation is contemplated. The nucleolus is the allocation that minimizes the maximum dissatisfaction. Put differently, it is impossible for any subset of creditors to get together and do better by themselves. Consistency requires this to be true regardless of the number of creditors in the subgroup; this method of dealing with fairness thus establishes stability among all creditors. On a more conceptual basis, if we pool our resources together as a separate group, do our shares represent the *same* "fair" division among us? An affirmative answer to the question characterizes the consistency property of any fair principle of division, which is absolutely necessary for any of us to accept a division as a fair allocation.

Aumann and Maschler mentioned that in the bankruptcy problem, the half-way point of the claim is a "psychological watershed." When the creditor receives less than half of her claim, her mind tends to minimize the debt and is happy for whatever she can get. When the creditor gets more than half her claim, her focus is on the entire debt and she concentrates on her loss. Importantly, "it is socially unjust for different creditors to be on opposite sides of the watershed."[54] It would be most upsetting to one creditor if she loses most of her claim while another creditor gets most of hers. The explicit tabulations of the allocation patterns across all estate values in Table 6 clearly shows that every creditor gains no more than half her individual debt when the estate is less than half of total debt. Although it is not readily seen from the expressions in the Table, it is easy to confirm that each creditor loses no more than half of her individual debt when the estate is greater than half of total debt.

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[51] In this case, her reference point is the *de facto* concession, the nonnegative amount conceded to her by the other creditor.

[52] Here, her reference point is the *de facto* claim: the claim itself or the estate if her claim exceeds the estate.

[53] Only the first of these three observations was known in the literature. This study identifies the second and third ways of viewing the CCC division outcomes.

[54] See Aumann and Maschler (1985, p.205).

Other than characterizing a CCC allocation through equal sharing of *de facto* contested claim, the fact that a CCC division can also be described as fulfilling equal sharing of *de facto* total loss shows how thoroughly equal the treatment of the bankruptcy division is. Whether one views what is there or looks at what is missing, creditors always share equally. We suggest that these distinguishing features of the CCC principle may serve as an adequate interpretation of the *pari passu* provision commonly incorporated in sovereign debt agreements.

Pari passu means "in equal step" or "on equal basis" in Latin; the provision makes its odd appearance in the sovereign debt instrument in the last century without anyone having a clear understanding of the meaning of the clause. The window-dressing provision sat in the loan documents for close to 100 years until Elliott Associates enlisted it as the legal weapon in the Brussels court to attach Peruvian payment streams through a Belgian clearinghouse to the Brady bond-holders.[55] As the precise meaning of *pari passu* was not readily clear, the holdout fund had to attribute a meaning to the phrase. They proposed that *pari passu* means proportional payment assignments. This was the first successful legal case again a sovereign country and the *ex parte* case became the cornerstone of other similar cases to come in the last fifteen years.

We propose the CCC allocation as a good contender to represent *pari passu*. CCC divisions incorporate equality every step of the way, gain or loss.[56] Between any pair of creditors, beyond the concessions from the other creditor, every additional amount awarded to the pair is split equally between the pair – this is clearly in equal step. Unlike the proportionality interpretation of *pari passu* which treats every dollar of debt equally, the CCC allocation treats every creditor equally – on equal basis. Thus, we suggest that applying the CCC principle to sort out the bankruptcy problem in general deserves to be seriously considered.

In their article presenting the answer to the age-old Talmudic bankruptcy puzzles, Aumann and Maschler expected their research to be of interest in the study of Talmud and in game theory. This paper aims to bring attention to the CCC division in the realm of bankruptcy in general and international sovereign debt default issues in particular. We believe that it is time for the CCC allocation to appear in mainstream discussion of bankruptcy divisions.

The in-depth study of finding the CCC allocation through equal sharing of *de facto* total loss brings some issues to the surface when compared to the proportional allocation commonly employed in bankruptcy situations. The CCC principle considers the part of the claim beyond the entire estate irrelevant, while the proportional principle takes the entire claim as relevant, including the portion that exceeds the entire estate. The opportunity to obtain partial repayment as well as the burden to bear the short-fall are both split equally between creditors in a CCC

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[55] *Elliott Assocs., L.P.*, General Docket No. 2000/QR/92 (Court of Appeals of Brussels, 8th Chamber, 26th Sept. 2000)

[56] Fon (2016*b*) argues this point in more detail.

division; this demonstrates the fact that each creditor counts equally. This is very different from the proportional allocation treatment in which creditors do not count equally and only the amounts of the claims matter. The CCC principle takes each creditor as equally important and their individual claims relevant under appropriate restrictions, while the proportional principle takes each dollar of each claim as equally important, independent of whom it belongs to and without restrictions. Thus, the CCC principle requires equal sharing of any contested claim (by any two creditors), while the proportional principle requires sharing of the entire claim proportionally.[57]

In general, whether the CCC allocation or the proportional allocation better divides the bankrupt estate depends on what one views as the appropriate criteria. Since the CCC allocation embedded in the millennia old Talmudic lessons was not well understood, the balance tilt towards the use of the proportional allocation. Now that the underlying principles behind the CCC allocation are better known, we should give strong consideration to CCC as an appropriate solution for a bankruptcy situation.

[57] See Fon (2016*a*) for more in-depth analysis.

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Appendix I: Equal Division of Contested Claim for Two Creditors

Without loss of generality, assume that the first claim is smaller than the second: $d_1 < d_2$. Given the bankrupt estate value E, the principle of equal division of contested claim specified by Aumann and Maschler (1985) provides the following shares to the two creditors, where $c_i^{\sim} = \max\{E - d_j, 0\}$: $s_1 = c_1^{\sim} + [E - c_2^{\sim} - c_1^{\sim}]/2 = [E - c_2^{\sim} + c_1^{\sim}]/2$, and $s_2 = c_2^{\sim} + [E - c_2^{\sim} - c_1^{\sim}]/2 = [E + c_2^{\sim} - c_1^{\sim}]/2$. While the general solutions for these divisions have been presented in Tables 2 and 3, this appendix provides more details. Special attention is paid to the behavior of the shares assigned to creditors as E grows with fixed claims $d_1 < d_2$. Boundary cases separating different categories of divisions are added. In Table A1, Column 1 details the relation between the estate and the two claims. Column 2 presents the individual shares in terms of the estate E and the claims; it also compares the creditors' shares as well as showing bounds for individual shares. In column 3, the estate is rewritten as a function of a claim or claims and a parameter, depending on the region, and creditor shares are presented in terms of the new expression.

Given estate E relative to the two claims d_1 and d_2 where $d_1 < d_2$	Shares allocated to the creditors according to the equal sharing of contested claim principle	Share of Creditors expressed in terms of d_1, d_2 and a parameter
$E < d_1 < d_2$ ($\Rightarrow$ $E < (d_1+d_2)/2$)	$s_1 = s_2 = E/2$ ($\Rightarrow$ $s_1 < d_1/2$; $s_2 < d_1/2 < d_2/2$)	$s_1 = s_2 = \alpha$ where $E = 2\alpha$ ($0 < \alpha < d_1/2$)
$E = d_1 < d_2$ ($\Rightarrow$ $E = d_1 < (d_1+d_2)/2$)	$s_1 = s_2 = d_1/2$	$s_1 = s_2 = d_1/2$ where $E = d_1$
$d_1 < E < d_2$ and $E < (d_1+d_2)/2$	$s_1 = d_1/2$, $s_2 = E - d_1/2$ ($\Rightarrow$ $s_1 = d_1/2$; $d_1/2 < s_2 < d_2/2$)	$s_1 = d_1/2$, $s_2 = d_1/2 + \beta$ where $E = d_1 + \beta$ ($0 < \beta < d_2/2 - d_1/2$)
$d_1 < E < d_2$ and $E = (d_1+d_2)/2$	$s_1 = d_1/2$ and $s_2 = d_2/2$	$s_1 = d_1/2$, $s_2 = d_2/2$ where $E = (d_1+d_2)/2$
$d_1 < E < d_2$ and $(d_1+d_2)/2 < E$	$s_1 = d_1/2$, $s_2 = E - d_1/2$ ($\Rightarrow$ $s_1 = d_1/2$; $d_2/2 < s_2 < d_2 - d_1/2$)	$s_1 = d_1/2$, $s_2 = d_2/2 + \beta'$ where $E = (d_1+d_2)/2 + \beta'$ ($0 < \beta' < d_2/2 - d_1/2$)
$d_1 < E = d_2$ and $(d_1+d_2)/2 < E$	$s_1 = d_1/2$ and $s_2 = d_2/2+(d_2-d_1)/2$	$s_1 = d_1/2$, $s_2 = d_2 - d_1/2$ where $E = d_2$
$d_1 < d_2 < E$ and $E < (d_1+d_2)$ ($\Rightarrow$ $(d_1+d_2)/2 < E < (d_1+d_2)$)	$s_1 = (E+d_1-d_2)/2$, $s_2 = (E-d_1+d_2)/2$ ($\Rightarrow$ $d_1/2 < s_1 < d_1$; $d_2/2 < d_2 - d_1/2 < s_2 < d_2$)	$s_1 = d_1/2 + \alpha'$, $s_2 = d_2 - d_1/2 + \alpha'$ where $E = d_2 + 2\alpha'$ ($0 < \alpha' < d_1/2$)
$d_1 < d_2 < E$ and $E = (d_1+d_2)$	$s_1 = d_1$ and $s_2 = d_2$	$s_1 = d_1$, $s_2 = d_2$ where $E = (d_1+d_2)$

Table A1. Shares According to the Principle of Equal Division of Contested Claim **($d_1 < d_2$)**

Column 3 in Table A1 is built to help sort out which cases are relevant and make it easier to discern the bankruptcy solution by reformatting and combining the first two columns. In particular, α in the first row is defined to equal $E/2$, or, $E = 2\alpha$, which implies that $\alpha<d_1/2$. The second row is equivalent to the boundary case where $\alpha=d_1/2$. (Boundary cases are important to sort out where a certain numerical case falls among all possible scenarios.) In the third row, since $d_1/2<s_2$, s_2 is redefined to be $s_2=(d_1/2+\beta)$, where $\beta < d_2/2–d_1/2$ because $s_2<d_2/2$. Row four corresponds to the case in which E is half the total claims and $\beta=d_2/2–d_1/2$. In the fifth row, $d_2/2<s_2$, s_2 is redefined to be $s_2=(d_2/2+\beta')$, where $\beta'<d_2/2–d_1/2$ because $s_2<d_2–d_1/2$. The sixth row is the boundary case in which $\beta'=d_2/2–d_1/2$. In the seventh row, since $d_1/2< s_1$, s_1 is redefined to be $s_1=(d_1/2+\alpha')$, where $\alpha'< d_1/2$ because $s_1<d_1$. With the introduction of α', s_2 can be rewritten as $s_2 = (E–d_1+d_2)/2 = (E+d_1–d_2–2d_1+2d_2)/2 = s_1–d_1+d_2 = (d_1/2+\alpha')–d_1+d_2 = d_2–d_1/2+\alpha'$. The last row is included for completeness, as it is the limit of α' when α' approaches $d_1/2$; it is not related to the bankruptcy problem. The last column of Table A1 provides the easiest way to observe how different division cases are separated. It is regrouped in Table 5 in another tabular form for ease of extension to three creditors and beyond.

Appendix II: CCC allocations for Three Creditors

To facilitate the proof of a 3-creditor CCC allocation, it is convenient to rewrite the restrictions on E and the corresponding allocations for different scenarios found in the 2-creditor case. This is because consistency is required under the CCC principle. This means that the "estate" must be repeatedly applied to the jointly awarded amount for any subset of two creditors before it is clear what type of award solution is appropriate in the current setting. To do this, we first rewrite the solutions in Table A1 as follows.

1) [Rows 1–2] If $E \le d_1 \le d_2$, $s_1 = s_2 = \alpha$ ($\le d_1/2$).
2) [Rows 3–6] If $d_1 \le E \le d_2$, $s_1 = d_1/2$, $s_2 = d_1/2+\beta$ ($\le d_2 - d_1/2$); and $\beta \le d_2 - d_1$.
   Note that we combine the allocations from rows 3 to 6 because the share given to the lesser creditor is constant at $d_1/2$. In doing so, the share to the greater creditor is written uniformly as $d_1/2+\beta$. Since $s_1+s_2 = E \le d_2$, this means that $\beta \le d_2 - d_1$.
3) [Rows 7–8] If $d_1 \le d_2 \le E$, $s_1 = d_1/2+\alpha'$ ($\ge d_1/2$), $s_2 = d_2 - d_1/2+\alpha'$ ($\ge d_2 - d_1/2$)

Rewrite the above three groups of solutions to facilitate our understanding of the basic assignment schedules for any two creditors i and j. Let their joint allocation be E^{ij}. Presenting the above divisions in more general form in terms of i and j, we have the following:

Case A. If $E^{ij} \le d_i \le d_j$, $s_i = s_j = \alpha$ ($\le d_i/2$).
Case B. If $d_i \le E^{ij} \le d_j$, $s_i = d_i/2$, $s_j = d_i/2+\beta$ ($\le d_j - d_i/2$), where $\beta \le d_j - d_i$.
Case C. If $d_i \le d_j \le E^{ij}$, $s_i = d_i/2+\alpha'$ ($\ge d_i/2$), $s_j = d_j - d_i/2+\alpha'$ ($\ge d_j - d_i/2$).

In general, we assume that the three creditors are ranked according to the size of their claims; their claims satisfy $d_1 \le d_2 \le d_3$. (This ranking is done for convenience.) From the results of the case of two creditors, we know that the allocations to the creditors are order preserving: $s_1 \le s_2 \le s_3$. The joint award assigned to any two creditors i and j is $E^{ij} = s_i + s_j$. Thus $s_1 + s_2 \le s_1 + s_3 \le s_2 + s_3$ imply that $E^{12} \le E^{13} \le E^{23}$.

Without loss of generality, we prove the different bankruptcy allocations for three creditors assuming $d_1 < d_2 < d_3$. When we compile the results in Table A2, which follows the proof, we include all boundary cases. While the proof focuses on how the relation between the aggregate sum available to any two creditors and their claims characterizes individual repayments under bankruptcy, the converse proof from individual repayments to the aggregate sum-and-claims relation is straightforward. That is, the entries in Table A2 are complete characterizations under different values of the estate and claims on the estate.

I. Assume $E^{12} \le d_1 < d_2$ ($< d_3$).
   Given $E^{12} \le d_1$, case A implies that $s_1 = s_2 \equiv \alpha \le d_1/2$. Assume that $d_2 \le E^{23}$; then cases B and C imply that $s_2 \ge d_2/2$. This contradicts $s_2 \le d_1/2$ since $d_1 < d_2$. Thus $E^{23} < d_2$ must hold in this case, and case A implies that $s_2 = s_3$, and $s_3 = \alpha$ also. To conclude:

(When $\alpha < d_1/2$) $s_1=s_2=s_3=\alpha$; $E=\Sigma s_i=3\alpha$. (Table A2 column 1)

(When $\alpha = d_1/2$) $s_1=s_2=s_3=d_1/2$; $E=\Sigma s_i=3d_1/2$. (Table A2 column 2)

II. Assume $d_1 < E^{12} \leq d_2$ and $E^{23} \leq d_2$.

When $d_1 < E^{12} \leq d_2$, case B implies that $s_1 = d_1/2$, $s_2 = d_1/2+\beta$, where $\beta\leq d_2-d_1$. With $E^{23} \leq d_2 < d_3$, case A implies that $s_2 = s_3$. Thus $s_3 = d_1/2+\beta$ as well, and $E^{23} \leq d_2$ implies that $d_1+2\beta\leq d_2$, or, $\beta\leq(d_2-d_1)/2$. (Note that $E^{12} = E^{13}$ in this case.) To conclude:

(When $\beta<(d_2-d_1)/2$) $s_1=d_1/2$, $s_2=s_3= d_1/2+\beta$; $E=\Sigma s_i=3d_1/2+2\beta$. (Table A2 column 3)

(When $\beta=(d_2-d_1)/2$) $s_1=d_1/2$, $s_2=s_3= d_2/2$; $E=\Sigma s_i=d_1/2+d_2$. (Table A2 column 4)

III. Assume $d_1 < E^{12} \leq d_2$ and $d_2 < E^{23} \leq d_3$.

Again $d_1 < E^{12} \leq d_2$ implies that $s_1=d_1/2$, $s_2=d_1/2+\beta$, where $\beta\leq d_2-d_1$. Likewise, $d_2 < E^{23} \leq d_3$ implies that $s_2=d_2/2$, $s_3=d_2/2+\gamma$, where $\gamma\leq d_3-d_2$. (Since the assigned amount to creditor 2 is unique, $d_1/2+\beta = d_2/2$ holds, and $\beta=(d_2-d_1)/2$ is implied.) What happens to the allocations depends on the value of γ.

a) (When $\gamma<(d_3-d_2)/2$) $s_1=d_1/2$, $s_2=d_2/2$, $s_3=d_2/2+\gamma$; $E=\Sigma s_i=d_1/2+d_2+\gamma$. (Table A2 column 5)

b) (When $\gamma=(d_3-d_2)/2$) $s_1=d_1/2$, $s_2=d_2/2$, $s_3=d_3/2$; $E=\Sigma s_i= (d_1+d_2+d_3)/2$. (Table A2 column 6)

c) When $(d_3-d_2)/2<\gamma< d_3-d_2$: Here $s_3=d_2/2+\gamma >d_3/2$. Define $s_3=d_3/2+\gamma'$. Since $E^{23} \leq d_3$, $s_2+s_3= d_2/2+d_3/2+\gamma'\leq d_3$ implies that $\gamma'\leq(d_3-d_2)/2$ holds. To conclude:

(When $\gamma'\leq(d_3-d_2)/2$) $s_1=d_1/2$, $s_2=d_2/2$, $s_3=d_3/2+\gamma'$; $E = (d_1+d_2+d_3)/2+\gamma'$. (Table A2 column 7)

d) (When $\gamma'=d_3-d_2$) $s_1=d_1/2$, $s_2=d_2/2$, $s_3=d_3-d_2/2$; $E = d_1/2+d_3$. (Table A2 column 8)

IV. Assume $d_1 < E^{12} \leq d_2$ and $d_3 < E^{23}$.

As usual, $d_1 < E^{12} \leq d_2$ implies that $s_1=d_1/2$, $s_2=d_1/2+\beta$, where $\beta\leq d_2-d_1$. ($d_2 <$) $d_3 < E^{23}$ and case C imply that $s_2=d_2/2+\beta'$, $s_3=d_3-d_2/2+\beta'$.

First, assume that ($d_1 <$) $d_3 < E^{13}$. Then case C implies that $s_1 = d_1/2+\beta''$, $s_3=d_3-d_1/2+\beta''$. Since $s_1=d_1/2$ also, $\beta''=0$; and $s_3=d_3-d_1/2$. This implies that $E^{13}= s_1+s_3 = d_3$, a contradiction to the assumption that $d_3 < E^{23}$.

Thus, ($d_1 <$) $E^{13} \leq d_3$ must hold. Case B implies that $s_1 = d_1/2$, $s_3 = d_1/2+\beta''$ where $\beta''\leq d_3-d_1$. The allocation amount for creditor 3 should be unique: $s_3=d_3-d_2/2+\beta'= d_1/2+\beta''$ implies that $\beta''= d_3-d_1/2-d_2/2+\beta'$. Together with $\beta''\leq d_3-d_1$, we have $\beta'\leq (d_2-d_1)/2$ must hold. (Note that the uniqueness of s_2 implies that $\beta=(d_2-d_1)/2+\beta'$, which further implies that $\beta\leq d_2-d_1$, consistent with what is required.) To conclude:

(When $\beta'<(d_2-d_1)/2$) $s_1=d_1/2$, $s_2=d_2/2+\beta'$, $s_3=d_3-d_2/2+\beta'$; $E= d_3+d_1/2+2\beta'$. (Table A2 col 9)

(When $\beta'=(d_2-d_1)/2$) $s_1=d_1/2$, $s_2= d_2-d_1/2$, $s_3=d_3-d_1/2$; $E=d_2+d_3-d_1/2$. (Table A2 col 10)

V. Assume $d_1 < d_2 < E^{12}$ $(< E^{13} < E^{23})$.

From case C, $d_1 < d_2 < E^{12}$ implies that $s_1 = d_1/2+\alpha'$ and $s_2 = d_2-d_1/2+\alpha'$.

Now assume that $(d_1 <)\ E^{13} < d_3$. Then case B dictates that $s_1 = d_1/2$, $s_3 = d_1/2+\beta$, $\beta>0$. Under this assumption, uniqueness of s_1 implies that $\alpha' = 0$, and $s_1=d_1/2$ and $s_2=d_2-d_1/2$. Further assume that $d_2 < E^{23} < d_3$. Again case B provides that $s_2 = d_2/2$, $s_3 = d_2/2+\beta''$, $\beta''>0$. Uniqueness of s_2 then implies that $d_2-d_1/2 = d_2/2$ which further implies that $d_1 = d_2$. This is a contradiction. The remaining case requires the assumption $d_2 < d_3 < E^{23}$, where case C gives $s_2=d_2/2+\alpha''$ and $s_3=d_3-d_2/2+\alpha''$. Uniqueness of s_2 then implies that $\alpha''=(d_2-d_1)/2$, which further implies that $s_3=d_3-d_1/2$. This means that $E^{13}= s_1+s_3 = d_3$; this is also a contradiction. Combining the two cases considered, we conclude that $d_3 \le E^{13}$ must be true.

Thus, $(d_1 <)\ d_3 \le E^{13}$ must hold. Now case C provides that $s_1=d_1/2+\alpha''$ and $s_3=d_3-d_1/2+\alpha''$. Uniqueness of s_1 implies that $\alpha'' = \alpha'$ and the allocations are $s_1=d_1/2+\alpha'$, $s_2=d_2-d_1/2+\alpha'$, and $s_3=d_3-d_1/2+\alpha'$. The bankruptcy problem implies that the total allocation to all creditors is no larger than the total debt owed to all creditors: $s_1+s_2+s_3 \le d_1+d_2+d_3$. This requires that $\alpha' \le d_1/2$. (Strictly speaking, there is no bankruptcy problem when $\alpha'=d_1/2$.) To conclude:

(When $\alpha'<d_1/2$) $s_1=d_1/2+\alpha'$, $s_2=d_2-d_1/2+\alpha'$, $s_3=d_3-d_1/2+\alpha'$; $E=d_2+d_3-d_1/2+3\alpha'$. (Table A2 col 11)

(When $\alpha'=d_1/2$) $s_1=d_1$, $s_2=d_2$, and $s_3=d_3$; $E=d_1+d_2+d_3$. (Table A2 col 12)

We conclude that the CCC allocations for the case of three creditors are the following.

	1	2	3	4	5	6	7	8	9	10	11	12
E \ Claim	3α, ($\alpha\le d_1/2$)	$3d_1/2$	$3d_1/2+2\beta$, ($\beta\le(d_2-d_1)/2$)	$d_1/2+d_2$	$d_1/2+d_2+\gamma$, ($\gamma\le(d_3-d_2)/2$)	$(d_1+d_2+d_3)/2$	$(d_1+d_2+d_3)/2+\gamma'$, ($\gamma'\le(d_3-d_2)/2$)	$d_3+d_1/2$	$d_3+d_1/2+2\beta'$, ($\beta'\le(d_2-d_1)/2$)	$d_2+d_3-d_1/2$	$d_2+d_3-d_1/2+3\alpha'$, ($\alpha'\le d_1/2$)	$d_1+d_2+d_3$
d_1	α	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2+\alpha'$	d_1
d_2	α	$d_1/2$	$d_1/2+\beta$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2/2+\beta'$	$d_2-d_1/2$	$d_2-d_1/2+\alpha'$	d_2
d_3	α	$d_1/2$	$d_1/2+\beta$	$d_2/2$	$d_2/2+\gamma$	$d_3/2$	$d_3/2+\gamma'$	$d_3-d_2/2$	$d_3-d_2/2+\beta'$	$d_3-d_1/2$	$d_3-d_1/2+\alpha'$	d_3

Table A2. 3-Creditor Shares Under the CCC Principle **($d_1 < d_2 < d_3$)**

To cement our understanding of the symmetric feature of the CCC divisions as the value of E changes, and to honor the historical breakdown in highlighting gains before half of the total claim and losses after half of the total claim, in the following we offer a second presentation of Table A2 (also Table 6 in the text) where all shares are presented in terms of gains. Table A3 is reformatted from Table A2 by replacing α', β', and γ' by $d_1/2-\alpha$, $(d_2-d_1)/2-\beta$, and $(d_3-d_2)/2-\gamma$, respectively. It is easily confirmed that the parameters α, β, and γ appearing in columns 11, 9, 7 play similar roles to those parameters in columns 1, 3, and 5, although in earlier (later) columns the parameters represent additions (subtractions). In particular, parameter α in both columns 1 and 11 is restricted to $\alpha \le d_1/2$; likewise, $\beta \le (d_2 - d_1)/2$, and $\gamma \le (d_3 - d_2)/2$ must hold. Some cells in Table A3 are highlighted to make it easier to observe the allocation symmetries between equal-sharing of the gain when E is small and equal-sharing of the loss when E is large. Equal-sharing of the gain can be seen in columns 1, 3, and 5 (a degenerate case involving one creditor); and equal-sharing of the loss appears in columns 11, 9, and 7. The symmetry between allocations when E is small and when E is large can be readily observed from comparing columns 1 and 11; columns 3 and 9; and columns 5 and 7.

	1	2	3	4	5	6	7	8	9	10	11	12
E / d_i	3α, ($\alpha \le d_1/2$)	$3d_1/2$	$3d_1/2+2\beta$, ($\beta \le (d_2-d_1)/2$)	$d_1/2+d_2$	$d_1/2+d_2+\gamma$, ($\gamma \le (d_3-d_2)/2$)	$(d_1+d_2+d_3)/2$	$\Sigma d_i-(d_1/2+d_2+\gamma)$, ($\gamma \le (d_3-d_2)/2$)	$\Sigma d_i-(d_1/2+d_2)$	$\Sigma d_i-(3d_1/2+2\beta)$, ($\beta \le (d_2-d_1)/2$)	$\Sigma d_i-3d_1/2$	$\Sigma d_i-3\alpha$, ($\alpha \le d_1/2$)	$d_1+d_2+d_3=\Sigma d_i$
d_1	α	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1/2$	$d_1-\alpha$	d_1
d_2	α	$d_1/2$	$d_1/2+\beta$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2/2$	$d_2-d_1/2-\beta$	$d_2-d_1/2$	$d_2-\alpha$	d_2
d_3	α	$d_1/2$	$d_1/2+\beta$	$d_2/2$	$d_2/2+\gamma$	$d_3/2$	$d_3-d_2/2-\gamma$	$d_3-d_2/2$	$d_3-d_1/2-\beta$	$d_3-d_1/2$	$d_3-\alpha$	d_3

Table A3. 3-Creditor Shares Under CCC Principle – Alternative Presentation **($d_1<d_2<d_3$)**